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MEDIAN ALGEBRA

BY

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ABSTRACT. A study of algebras with a ternary operation (x, y, z) satisfying some identities, equivalent to embeddability in a lattice with (x, y, z) realized as, simultaneously, $(x \wedge (y \vee z)) \vee (y \wedge z)$ and $(x \vee (y \wedge z)) \wedge (y \vee z)$. This is weaker than embeddability in a modular lattice, where those expressions coincide for all x, y, and z, but much of the theory survives the extension. For actual embedding in a modular lattice, some necessary conditions are found, and the investigation is carried much further in a special, geometrically described class of examples ("2-*cells*"). In distributive lattices (x, y, z) reduces to the median $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, previously studied by G. Birkhoff and S. Kiss. It is shown that Birkhoff and Kiss found a basis for the laws; indeed, their algebras are embeddable in distributive lattices, i.e. in powers of the 2-element lattice. Their theory is much further developed and is connected into an explicit Pontrjagin-type duality.

Introduction. This paper began in a study of the ternary operation *median* in distributive lattices: that is, (x, y, z) defined as $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, or by the dual formula. This subject was broached in 1947 by G. Birkhoff and S. Kiss, who found [3] four identities true of this operation and showed that a ternary algebra which satisfies those identities and also has two suitable elements 0, 1 ("suitable" meaning $(0, x, 1) \equiv x$) is a distributive lattice.

Here the algebras of Birkhoff and Kiss are called *symmetric media*. It is shown that they are all embeddable in distributive lattices. Equivalently, they are embeddable in powers of $\mathbf{2} = \{0, 1\}$. Or, each point of a symmetric medium M is an intersection of fibers $h^{-1}(0)$ of homomorphisms h: $M \to \mathbf{2}$–*halfspaces*. More fully, a subset I of M is an intersection of halfspaces if and only if (x, y, z) is in I whenever y and z are in I. Such sets are called *ideals*.

Finite symmetric media (and a larger class "connected" by finite ideals) are complexes of cubes constructed as follows. One can amalgamate two symmetric media M_1, M_2, along an isomorphism of two nonempty ideals $I_k \subset M_k$, getting $M = M_1 \cup (M_2 - I_2)$. In the amalgam, a subset J is an ideal if and only if both $J \cap M_k$ are ideals and, if both are nonempty, each meets I_k. The cubes $\mathbf{2}^n$, and infinite analogs, are not amalgams; the indicated media are made from these by amalgamation.

Some more geometry: the smallest ideal containing two points x, y, is called the *interval* $[xy]$. Theorem. *The smallest ideal containing two nonempty ideals I, J, is the*

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union of all $[xy]$, $x \in I$ *and* $y \in J$. Some functorial semantics (if the term is unfamiliar, try "model theory"): the known categorical duality between distributive lattices and certain compact partially ordered sets "extends" to a duality between symmetric media and certain compact posets with additional structure. The additional structure is just a unary order-reversing operation $'$. In a sense, this confirms that the forgetful functor from lattices to media "forgets only up-down". (There is another result to the same effect, 6.17 below.)

The principal subject of this paper is a nonsymmetric generalization. There are reasons for calling it the algebra of projections rather than the algebra of medians, which seem quite insufficient in view of all the other projections in mathematics. But the first nonsymmetric step is to the operation $(x, y, z) = (x \wedge (y \vee z)) \vee (y \wedge z)$ in modular lattices; this is the projection of x in the smallest interval that contains y and z. (In the distributive case this formula gives the median.) It is important in modular lattices (and characteristic of them) that (x, y, z) coincides with $[x, y, z] = (x \vee (y \wedge z)) \wedge (y \vee z)$.

The laws of the median (x, y, z) in subsets of modular lattices closed under median are not determined. The basis of the paper is a determination of the laws of median for subsets M of lattices which are closed under (x, y, z) and on which $[x, y, z] = (x, y, z)$: five identities. Indeed, if that were the only basis, the work would be very difficult, since the identities are fairly awkward to use. But just as in lattices one relies more on the partial order than on the formal laws of $\vee$ and $\wedge$, so here–in *media*–there is a ternary relation of betweenness sufficient and more convenient for describing the structure. We say that x is *between* y and z if $x \in [yz]$, the interval, defined as in the symmetric case.

Convenient or not, is there enough structure in these subsets of lattices to justify the effort? It seems so. There are preliminary difficulties with Jordan-Hölder type theorems, for (1) projection $x \mapsto (x, a, b)$ does not take intervals to intervals even in submedia of modular lattices, and (2) even when it maps an interval bijectively to $[ab]$ the map need not be isomorphic. Both difficulties vanish for minimal nonsingleton intervals, which are called *edges*. A *path* from x_0 to x_n is defined as a sequence of intervals $[x_{i-1}x_i]$ joining them, with $[x_0x_i] \cap [x_ix_n] = \{x_i\}$ and $[x_0x_i]$ containing $x_1, \dots, x_{i-1}$. In an *edge-connected* medium (definition obvious), any two paths from a to b have refinements related by a unique bijection φ such that intervals I and $\varphi(I)$ project to each other isomorphically. By the way, this result, though widely known for modular lattices and substantially due to Zassenhaus **[10]** and Birkhoff **[2]**, seems not to be in print and not to have a name. It is much sharper than the Jordan-Hölder theorem (which admits composites of projections and has no uniqueness clause); call it a Zassenhaus-Birkhoff theorem.

The median (x, y, z) in a medium may be characterized as the only point between y and z (i.e. in $[yz]$) which is between x and each point of $[yz]$. This is a "nearest point" in a quite strong sense. Any subset S of a medium in which each point x has a nearest point in that sense, $(x; S)$, is called a *Čebyšev set*. Theorem. *If two Čebyšev sets* S, T, *have a common point then* $S \cap T$ *is Čebyšev, the projections* $(x; S)$ *and* $(x; T)$ *commute, and their composite is* $(x; S \cap T)$. Theorem. *A subset* *of an edge-connected medium is a Čebyšev set if and only if it contains every path of edges between its points.*

With any edge-connected medium M there is associated a *reduced* medium M_0 on the same points, whose ideals are the Čebyšev sets of M. The Jordan-Hölder theory of M is substantially that of M_0; in other words, the principal results for general media may be considered as "really" about reduced media. Reduced media are definitely better behaved. For instance, projections of intervals in intervals are Čebyšev ideals, and bijective projections are isomorphic. But reduced media are by no means all embeddable in modular lattices.

The status of the modular-lattice embedding problem is this. We find two independent identities which are necessary. Perhaps they are sufficient. A special study is made of *2-cells*, defined as media which are intervals $[ac] = [bd]$ containing a cycle of edges *abcd* of which any two adjacent ones form a path. The theory of 2-cells looks very much like the theory of projective planes (if one allows for its youth). There are partial 2-cells, free 2-cells, and dual 2-cells. One of the "modular" identities is satisfied by all 2-cells (and n-cells, defined in the same style). Let us digress to that identity, *tautness*, which is:

$$((x, y, z), y, (z, (x, y, z), y)) = (x, y, z).$$

It implies the results on projections noted for reduced media, and thus a Zassenhaus-Birkhoff theorem. The second identity is much longer (in six variables) but comprehensible; it is the *axiom of Pasch*. Rectangles being defined like 2-cells except that the sides need not be edges, the Pasch axiom says that two intervals that cross a rectangle in the two directions must meet.

Returning to 2-cells, those that are known to be embeddable in modular lattices are of three types. (1) Bipartite graphs, with a set of points $A \cup B$ and the intervals $[xy] = \{x, y\}$ if $\{x, y\}$ meets A and B, $[xy] = A \cup B$ otherwise. (2) Products of edges $E \times F$. (3) One 2-cell for each commutative field k, which is embeddable in PG(3, k) as the set of fixed lines of a suitable autoduality (a null system **[1]**). Types (1) and (2) are dual (by 2-cell duality which exchanges points and edges). The dual of a type (3) 2-cell is not Paschian unless k has characteristic 2. There are Paschian 2-cells for which embeddability in a modular lattice is unsettled; the smallest has 27 points.

One sees that there are a great many 2-cells and little structure theory so far. In n-cells, by the way, are included $(n - 1)$-dimensional projective geometries. (For $n = 2$, that is only some of the bipartite graphs.) Note this much: every n-cell is taut and reduced, and any two of its points belong to one of the defining 2^n-tuples of vertices.

Matters are simpler in media whose projections $x \mapsto (x, a, b)$ preserve betweenness, far simpler. The projections are then homomorphic, so this is a variety of media called *isotropic*. They are just the subdirect products of edges. Accordingly much goes the same as in symmetric media. All finite edges are projective in this variety, and this leads to a determination of all subvarieties. (There is one for each $n = 2, 3, \ldots,$ defined by the laws of an n-point edge.) An approximate analysis of isotropy: an edge-connected medium is isotropic if and only if it is reduced and projective isomorphism is transitive.

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1. General media. The algebraic axioms defining a medium will be unmentioned through most of the paper, after we have a geometric translation of them. To make them a little less colorless, let us think of (x, y, z) as the "nearest point to x between y and z" from the beginning, and observe that in many metric spaces, too, there is such an operation. The axioms are

(1) $(x, y, z) = (x, z, y)$,

(2) $(x, x, y) = x$,

(3) $((x, (y, u, v), (z, u, v)), u, v) = (x, (y, u, v), (z, u, v))$,

(4a) $(x, (x, u, v), (y, u, v)) = (x, u, v)$,

(4b) $(x, (x, u, v), (v, x, (x, u, v))) = (v, x, (x, u, v))$,

(5) $((x, y, z), x, y) = (x, y, z)$.

Ternary algebras assumed only to satisfy (1)–(4b) will be the subject of a few remarks and will be called *foot algebras* (the foot, of x in $[yz]$, being (x, y, z)). In a foot algebra, the *interval* $[uv]$ is the set of all x such that $(x, u, v) = x$. Its elements will be said to lie *between* u and v. It is (i) the same as the set of all elements of the form (t, u, v). For if x is in $[uv]$, it is (x, u, v). Conversely, $((t, u, v), u, v) = ((t, (u, u, v), (v, u, v)), u, v)$ by (2) and (1), so by (3), $(t, u, v) \in [uv]$.

A set is an *ideal* if it contains all points between its points. (ii) Intervals $[uv]$ are ideals; for three uses of (i) reduce (3) to "x^*, between y^* and z^* in $[uv]$, is in $[uv]$".

Axioms (1) and (2) of course say $[yz] = [zy] \supset \{y, z\}$. (4a), read casually: the nearest point x^* to x in $[uv]$ is nearest also in $[x^*y^*]$ if $y^* \in [uv]$. But then (iii) x^* is also nearest to x in $[st]$ whenever $x^* \in [st] \subset [uv]$. For let $x^{**} = (x, s, t)$; (x, x^*, x^{**}) must be x^* since $x^{**} \in [uv]$ and must be x^{**} since $x^* \in [st]$. Axiom (4b) says: construct $z = (x, u, v)$; come back from an endpoint v to the nearest point p in $[xz]$; then p must be nearest to x in $[pz]$. Axiom (5) read casually: (x, y, z) is between x and either endpoint. But by (4a) any $y^* \in [yz]$ can substitute for the endpoint; (iv) (x, y, z) is between x and each point of $[yz]$.

In the proof of Theorem 1.3 we shall use the fact that Axioms (1), (2), (3), (5) suffice for: $(x, y, z) \in [xy]$. And further, (v) still without (4a), (4b), $(x, y, z) \in [uv]$ if two of x, y, and z are in $[uv]$.

1.1. *A finite number of medium ideals have a common point if every pair of them do.*

Proof. If ideals A, B, C have points γ, α, β in their pairwise intersections, then (α, β, γ) is in A since β and γ are, in B since α and γ are, and in C since α and β are. The case of n ideals $I_1, \ldots, I_n$ reduces to the cases of $n - 1$ and 3 by considering I_1, I_2, and $I_3 \cap \ldots \cap I_n$.

1.1 says that the ideals of a medium have Helly number 2; in other words, the medium has Helly dimension 1. Conversely:

1.2. *A foot algebra which has Helly dimension* 1 *is a medium.*

PROOF. We need $t = (x, y, z)$ to be in $[xy]$. Pairwise, $[xt]$, $[ty]$, $[xy]$ have common endpoints, so the three intervals have a common point c. Then $t = (x, t, c)$ by (4a); that is $(x, (x, t, c), (c, x, (x, t, c)))$ by (4a) again, and that is (by (4b)) $(c, x, (x, t, c)) = c \in [xy]$.

If one wants the most convenient algebraic axioms for media, one will come closer by removing (4a) and (4b) and substituting

(4)
$$((y, u, v), x, (x, u, v)) = (x, u, v),$$

which, we shall see, is equivalent given the other axioms. But on the basis (1)–(3), it is much stronger. (vi) Every strictly convex Banach space is a foot algebra, but it satisfies (4) if and only if perpendicularity is symmetric. (Routine proof omitted.)

(4) implies

$$\begin{aligned}(x, (x, u, v), (y, u, v)) &= ((x, (x, u, v), x), ((y, u, v), (x, u, v), x), (y, u, v)) \\ &= ((y, u, v), (x, u, v), x) = (x, u, v),\end{aligned}$$

which is (4a). Also (4) implies

$$(y, x, x) = ((y, x, x), x, x) = ((y, x, x), (x, x, x), x) = (x, x, x) = x;$$

singletons are intervals. Then

$$\begin{aligned}(x, (x, u, v), (v, x, (x, u, v))) &= (x, (x, u, v), ((v, u, v), (x, u, v), x)) \\ &= (x, (x, u, v), (x, u, v)) = (x, u, v),\end{aligned}$$

which is (4b).

Returning to Axioms (1)–(4b), still

$$\begin{aligned}(y, x, x) = (y, x, (x, x, y)) &= (x, (x, x, y), (y, x, (x, x, y))) \\ &= (x, x, (y, x, (x, x, y))) = x.\end{aligned}$$

(vii) Singletons *are* intervals.

1.3. THEOREM. *The set $\mathcal{I}$ of ideals of a medium is closed under intersection, includes each set that contains elements of $\mathcal{I}$ containing all pairs of its distinct points, and satisfies the following condition. Let $|uv|$ denote the smallest element of $\mathcal{I}$ containing $\{u, v\}$. For each x, y, z, there is a unique p in $|yz|$, such that for all q in $|yz|$, $p \in |xq|$.*

Any set of subsets of a set having those properties is the set of ideals of a unique medium.

PROOF. The ideals of a medium satisfy the preliminary conditions by definition and (vii). Since $[uv]$ is an ideal containing u, v, and contained in all such ideals, $[uv] = |uv|$. (5) says $(x, u, v) \in [qx]$ if $q \in [uv]$. Suppose $z \in [uv]$ also has that property. Then $(x, u, v) = (x, y, z)$ where $y = (x, u, v)$. Since $(x, y, z) \in [uv]$, z is in $[x(x, y, z)]$, $(z, x, (x, y, z)) = z$. Also, by (4b), $(z, x, (x, y, z)) = (x, (x, y, z), (z, x, (x, y, z)))$; this is $(x, (x, y, z), z) = (x, y, z)$ by (4a), which is the required uniqueness. This shows also that the set $\mathcal{I}$ determines the operation.

Given a family $\mathcal{I}$ with these properties, define (x, y, z) as the nearest point of

$|yz|$ to x. Axioms (1) and (2) hold since we have a closure operation. From the definition, $x \in |uv|$ if and only if x has the form (t, u, v), and $|uv| = [uv]$. So (3) holds since we have a closure operation. Axiom (5) states the defining property of (x, y, z). Now the left side of (4), $l = ((y, u, v), x, (x, u, v))$, is between u and v (since Axioms (1), (2), (3), (5) hold). It is also between x and (x, u, v). For every q in $[uv]$, $[xq]$ contains (x, u, v), $[x(x, u, v)]$, and l. By uniqueness, (4) holds, hence (4a) and (4b). The set is a medium with this operation, and since $|uv| = [uv]$, $\mathcal{I}$ is the set of ideals, as required.

The convex sublattices of a modular lattice L satisfy the conditions of Theorem 1.3. The nearest point (x, u, v) is $p = (x \vee m) \wedge j$, where m is $u \wedge v$ and j is $u \vee v$. For if q is between u and v, which means between m and j in order, then $x \wedge q \leqslant x \wedge j \leqslant p$, and dually $x \vee q \geqslant p$. Thus p is a nearest point. But $[xp]$ touches $[mj]$ only at p. For a common point must be over $x \wedge p = x \wedge j$ and over m, hence over $(x \wedge j) \vee m$, which is p since L is modular; and dually.

(viii) The foregoing paragraph applies more generally to any subset M of a lattice such that $(x \vee (u \wedge v)) \wedge (u \vee v) = (x \wedge (u \vee v)) \vee (u \wedge v) \in M$ for all x, u, v in M; that equation is the only modularity we used. A medium of this form, or any medium isomorphic to it, will be said to be *embedded in a lattice*.

1.4. THEOREM. *Every medium can be embedded in a lattice.*

PROOF. Let M be a medium and $\mathcal{I}$ its set of ideals. For each subset $\mathcal{F}$ of $\mathcal{I}$ let $\mathcal{F}^*$ be the set of all $I \in \mathcal{I}$ that have nonempty intersection with every $J \in \mathcal{F}$. Evidently * is a Galois connection, so the subsets $\mathcal{F}^*$ of $\mathcal{I}$ form a complete lattice L. Map M to L by identifying each $x \in M$ with the set of all ideals containing x. For any $u, v \in M$, $u \wedge v$ is the set of ideals containing $\{u, v\}$, i.e. containing $[uv]$. Since * reflects L in M, $u \vee v$ is $(u \wedge v)^*$, the set of ideals meeting $[uv]$. Then $x \wedge (u \vee v)$ is the set of ideals containing x and a point of $[uv]$, i.e. containing $[x(x, u, v)]$; so $x \vee (u \wedge v)$ is the set of ideals meeting $[x(x, u, v)]$. To meet $[x(x, u, v)]$ and $[uv]$ is to contain (x, u, v), and with *, the proof is complete.

It seems worth mention that (ix) no nonmodular lattice is betweenness-isomorphic with a foot algebra. (Proof omitted.) There will not be another word about foot algebras.

We call a subset C of a medium M a *Čebyšev set* if it shares with intervals the property that for each point x there is a unique point $(x; C)$ of C which is between x and every point of C. The function $x \mapsto (x; C)$ is called *projection* upon C. It need not be homomorphic nor even betweenness-preserving. Even in the smallest nondistributive modular lattice M_1, which consists of 0, 1, and three incomparable elements a, b, c, projection upon $[0a]$ takes b and c to 0 but leaves fixed $a \in [bc]$. Also, M_1 has a submedium $\{0, a, b, c\}$ which has three Čebyšev sets that are not ideals, such as $\{0, a, b\}$.

Let us say that two (Čebyšev) sets *tick*, at p, if they have exactly one common point p.

1.5. *If an interval* $[ab]$ *ticks a Čebyšev set* C *at* p, *then* $p = (x; C)$ *for every* x *in* $[ab]$.

PROOF. $(x;\ C) \in [xp] \subset [ab]$, and p is the only available member of C.

1.6. *If two Čebyšev sets S, T have a common point, then for each x in S, $(x;\ T)$ is in $S \cap T$.*

PROOF. Let $x \in S$ have $(x;\ T) = t$ not in S. Let $(t;\ S) = s \neq t$. But $s \in [tx]$ which ticks T at t. So $t = (s;\ T)$ too. Let $p \in S \cap T$. Then (p, s, t) is not s, since $t = (s;\ T) \in [sp]$. Since $(p, s, t) \in [st]$ which ticks S at s, s is $((p, s, t);\ S)$. But $s \notin [(p, s, t)p]$, a contradiction.

It follows (by the way) that Čebyšev sets, though they need not be ideals, are submedia. More:

COROLLARY. *If $[xy]$ ticks $[yz]$, then every Čebyšev set containing $\{x, z\}$ contains y.*

For $(y;\ S)$ must be in $[xy] \cap [yz] = \{y\}$.

From 1.6, also, the nonempty intersection $S \cap T$ is a *relative Čebyšev set* in S, i.e. a Čebyšev set of the submedium S.

1.7. *A relative Čebyšev set U in a Čebyšev set S is Čebyšev, and $(x;\ U) = ((x;\ S);\ U)$.*

PROOF. First, a special case; suppose (I) that S is a Čebyšev ideal. For any point p, let $f = (p;\ S)$, $g = (f;\ U)$. For any point u of U, consider $\gamma = (u, f, g)$. Since $[fg]$ ticks U at g (since S is an ideal), $(\gamma;\ U) = g$ by 1.5. Then g is in $[u\gamma]$ which ticks $[fg]$; so $\gamma = g$. Now, knowing that $[ug]$ and $[fg]$ tick, $(f, u, g) = g$. Consider $\gamma^* = (p, u, g)$. Since $\gamma^* \in [ug] \subset S, f \in [p\gamma^*]$. So (f, u, g) is also γ^*, $\gamma^* = g$. The arbitrary point u caught $g = ((p;\ S);\ U)$ in $[pu]$; and since $[pg]$, $[ug]$ tick, u is not in $[pg]$ unless $u = g$. That is, g is $(p;\ U)$.

The general case, we remark, can be done like case (I) by extending the lemmas. Instead we shall extend all facts about media by producing a new medium M_S in which the given Čebyšev set S becomes an ideal. We define $|yz|$ to be $[yz] \cap S$ if y and z belong to S, $[yz]$ otherwise. We define $\mathfrak{J}$ as the set of all subsets of the set of points which contain $|yz|$ whenever they contain $\{y, z\}$. $\mathfrak{J}$ satisfies the preliminary conditions of Theorem 1.3 by definition, and our $|uv|$ is indeed the smallest element of $\mathfrak{J}$ containing $\{u, v\}$. Next consider y, z, not both in S, and any point x. The point (x, y, z) (the median in M) is a point p of $|yz| = [yz]$ such that $|xp|$ (and even $[xp]$) ticks $[yz]$. For q in $[yz]$, $|xq|$ could fail to contain p only if $\{x, q\} \subset S$ and $p \notin S$; but by 1.6, x in S and $[yz]$ meeting S in q imply $(x, y, z) \in S$. So p is the required point. There remains the case $\{y, z\} \subset S$, for which the nearest points $p = (x;\ |yz|)$ are given by 1.6 and case (I). Since $p \in S$ it continues to lie in $|xq| \supset [xq] \cap S$ for all $q \in |yz|$. So M_S is a medium, and S is an ideal in it.

Every Čebyšev set T of M, or relative Čebyšev set T in S, retains that character in M_S. Intervals $|x(x;\ T)|$ tick T because $[x(x;\ T)]$ does. For $|xt|$ to omit $(x;\ T)$ would require x, t in S, but $(x;\ T)$ not, impossible by 1.6 (and in the relative case, by $T \subset S$). Accordingly 1.7 in M follows from case (I) in M_S.

1.8. THEOREM. *A nonempty intersection of two Čebyšev sets is a Čebyšev set, the projections commute, and the composite is projection upon the intersection.*

1.9. *For any finite set $\mathcal{C}$ of Čebyšev sets of a medium M, the sets which are the intersection of an ideal and several elements of $\mathcal{C}$ constitute the ideals of a medium $M_{\mathcal{C}}$.*

Theorem 1.8, of course, follows from 1.6 and 1.7. And 1.9 follows by induction from the proof for a singleton $\{S\}$, since Čebyšev sets of M remains Čebyšev in M_S.

1.10. *A medium M_S has just the same Čebyšev sets as M.*

PROOF. It remains to show that a set D not Čebyšev in M cannot become Čebyšev. If it did, some point x not having a unique nearest point in D in the structure of M acquires such a point f in the geometry of M_S. Failure in M might mean $f \notin [xg]$ for some $g \in D$; but then $f \notin |xg|$ either. So f is a nearest point to x in D, already in the medium M; there must be another g in D with $[xg] = [xf]$. Since $g \notin |xf|$, $|xf| \neq [xf]$ and $x, f \in S$. Then by the Corollary to 1.6, the median in M, $(x, f, g) \in S$; since it is in $|xf|$, it is f. This means $[xf]$ ticks $[fg]$, a contradiction.

REMARK. By finite induction, 1.10 applies to media $M_{\mathcal{C}}$. For an infinite family $\mathcal{C}$ of Čebyšev sets, the obvious type of transfinite construction of $M_{\mathcal{C}}$ sometimes fails. (It may sometimes depend on the order of steps–I do not know.) Whenever it does work, 1.10 applies; just look at the first step making $|xf| \neq |xg|$.

Here is an example (x) where not all Čebyšev sets can be adjoined to the ideals. Let the points of M be x_j^i for $i = 1, 2, 3$ and for $j = 1, 2, \ldots$. Between x_m^i and x_n^i are the x_k^i with k between m and n; between x_m^i and x_n^j, $i \neq j$, $m \leqslant n$, are the x_r^i with $r \geqslant m$, the x_s^j with $s \geqslant n$, and the x_t^k, $k \neq i, j$, with $t \geqslant n + 1$. One may check that this is a medium and that three final segments of the x_i^1, the x_j^2, and the x_k^3 make a Čebyšev set. But allow them all, and (x_1^1, x_1^2, x_1^3) does not exist.

Now one naturally turns to finiteness conditions. First, some more negatives. (xi) Finitely spanned ideals need not be Čebyšev sets. There is a more basic reason for that. (xii) Whether x_0 belongs to the ideal spanned by x_1, x_2, x_3 is not an absolute relation among x_0, x_1, x_2, and x_3, i.e. it is not determined in the submedium they generate. The points x_i of a quadrilateral in a projective plane P, together with the empty subspace 0, form a medium M in which every set containing 0 is an ideal; so x_0 is not in the ideal $\{x_1, x_2, x_3, 0\}$. But in the lattice of all subspaces, P is between (P, x_1, x_2) and x_3, so x_1, x_2, x_3 do not lie in any proper ideal.

Therefore, in a free medium F on $\aleph_0$ generators $y_0, y_1, \ldots$ (in the smallest variety V containing some plane P), y_0 has no nearest point w in the ideal spanned by y_1, y_2, y_3. For w would be a word in finitely many generators $y_0, y_1, \ldots, y_n$. There is a homomorphism into the subspace lattice of P taking y_i to x_i for $i \leqslant 3$, y_j to x_3 for $3 < j \leqslant n$, and y_k to P for $k > n$. w goes into the submedium $\{x_1, x_2, x_3, 0\}$ (an ideal in the submedium M); so w was not between y_0 and $(y_{n+1}, y_3, (y_{n+1}, y_1, y_2))$. Thus (xi).

I do not know if a finitely spanned ideal in a finitely generated medium must be a Čebyšev ideal. (I would guess not; note, (x) is finitely generated. But in V?) In a modular lattice (medium), finitely spanned ideals are intervals. Even there, I do not know if the ideal join of two Čebyšev ideals need be Čebyšev.

We need geometric finiteness. Call an interval an *edge* if it is not a singleton but all its proper subintervals are.

1.11. *A nondegenerate interval which is not an edge is a union of nondegenerate proper subintervals.*

PROOF. If $[ab]$ has a proper subinterval $[cd]$ then any x in $[ab]$ is either in $[cd]$ or in nondegenerate proper $[x(x, c, d)]$.

1.12. *Projection upon a Čebyšev set takes an edge either to a single point or bijectively to an edge.*

PROOF. If the image is not a single point, the projection φ upon S takes the edge $[ab]$ to at least two points $\varphi(a) \neq \varphi(b)$. Then φ maps $[ab]$ injectively, for x in $[ab]$ is recovered as $(\varphi(x), a, b)$, and that because $[x\varphi(x)]$ contains x and $(\varphi(x), a, b)$ but not all of $[ab]$. Now if $[\varphi(a)\varphi(b)]$ were not an edge it would have a nondegenerate proper subinterval $[\varphi(a)m]$, and we may secure that $[m\varphi(b)]$ ticks $[\varphi(a)m]$. Then $c = (m, a, b)$ is different from a or different from b; we may choose notation so that $c \neq a$. Following φ with projection to $[m\varphi(b)]$, we have projection upon $T = S \cap [m\varphi(b)]$ taking a to $m \neq \varphi(b)$. So $[am]$ does not contain $\varphi(b)$, and hence not b. But it contains two points a, c, of the edge $[ab]$, a contradiction.

Thus any two points s, t of $\varphi([ab])$ lie in an edge. For three points s, t, u, the three edges, meeting in pairs, have a common point, and therefore coincide; $\varphi([ab])$ is in a single edge E. The projection of E to $[ab]$ is not constant, so it is an injection inverting φ.

We define two intervals $[ab]$, $[cd]$ to be *parallel* if projection from $[ab]$ upon $[cd]$ is bijective. (Then (xiii) projection upon $[ab]$ inverts it, since we have intervals $[x(x, c, d)]$ ticking both.) The intervals $[ab]$, $[cd]$, so named (note!), are *perspective* if $(a, c, d) = c$, $(b, c, d) = d$, $(c, a, b) = a$, and $(d, a, b) = b$. We may as well name the figure, intervals $[ab]$, $[bd]$, $[dc]$, $[ca]$, ticking in cyclic order at distinct vertices: a *rectangle*. Of course it also exhibits perspectivity of $[ac]$ and $[bd]$. We should also name the projections in these relationships, respectively *parallelism*, *perspectivity*. Since they are determined by the intervals, this is on the wordy side, but there arise also nonunique isomorphisms and composites of parallelisms. The latter, we call *transparallelisms*; their domain and codomain are *transparallel*.

Restating a special case of 1.12: perspective edges are parallel and isomorphic. In fact, this is true of two intervals if one of them is known to be an edge.

In modular lattices, simple calculations show that a lattice perspectivity is a medium perspectivity; and a medium perspectivity is a lattice projectivity, thus an isomorphism.

(xiv) In general media, perspective intervals need not be parallel. For this, hook up the lattice $\{1, 2, 3\}$ with the submedium $\{0, a, b, c\}$ of the five-element lattice M_1 so that the 22 ideals are nine trivial, seven subintervals of those two media, $\{1, a\}$, $\{2, 0\}$, $\{2, 0, b\}$, $\{3, c\}$, $\{1, 2, a, 0\}$, and $\{2, 3, 0, c\}$. And look.

(xv) Parallel intervals need not be isomorphic. Eight points 0, a, b, c, $0'$, a', b', c'; 26 ideals, ten trivial and four $\{x, x'\}$, three $\{x, x', 0, 0'\}$, 3 + 3 $\{x, 0\}$ and $\{x', 0'\}$, $\{a, b, 0\}$, $\{a, b, c, 0\}$, and $\{a', b', c', 0'\}$.

We define a *path* from a to b as a sequence of intervals $[x_{i-1}x_i]$, $i = 1, \ldots, n$, with $x_0 = a$ and $x_n = b$, such that for each i, $[x_0x_i]$ ticks $[x_ix_n]$, $x_j \in [x_0x_i]$ for $j < i$, and $x_j \in [x_ix_n]$ for $j > i$. There is redundancy in these conditions, which is fairly well cleared out by

1.13. *The concatenation of paths from a to b and from b to c, where $[ab]$ ticks $[bc]$, is a path.*

PROOF. For x_i in $[bc]$ (for instance), $[a(a, x_i, c)]$ contains $b = (a, b, c)$; it ticks $[x_ic]$, necessarily at $(b, x_i, c) = x_i$.

COROLLARY. *A sequence of edges $[x_{i-1}x_i]$ for $i = 1, \ldots, n$ is a path if $[x_ix_n]$ never contains x_{i-1}.*

PROOF. We may suppose the sequence for $i = 2, \ldots, n$ is a path. As $[x_1x_n]$ does not contain x_0, its intersection with $[x_0x_1]$ is a proper ideal in that edge; and the lemma applies.

We call a medium *artinian* if every descending sequence of intervals is finite. Points a, b are *edge-connected* if there is a (finite) set of edges whose union contains a and b and cannot be partitioned into two nonempty sets in one of which each of those edges is contained. (The points reachable from a by strings of edges successively ticking each other would be one member of such a partition if the complement were not empty.) The medium is *edge-connected* if all pairs of its points are.

1.14. *Artinian media are edge-connected.*

PROOF. Given two distinct points a, b, either $[ab]$ is an edge or it has a proper subinterval I containing a. By descent we may assume I is an edge. Put $x_1 = (b; I)$. By descent we may assume that x_1 and b are edge-connected, so a and b are.

1.15. *If points a, b are edge-connected then every path from a to b is refined by a path of edges.*

PROOF. If edges $[x_{i-1}x_i]$ for $i = 1, \ldots, m$ successively tick and join a to b, and intervals $[y_{j-1}y_j]$ for $j = 1, \ldots, n$ make a path from a to b, projection upon $[y_0y_1]$ takes the $[x_{i-1}x_i]$ to singletons and edges beginning at $y_0 = a$ and ending at y_1. Ignoring the singletons, some k edges $[y_0w]$, etc., join y_0 to y_1. Now $(y_1, y_0, w) = z \neq y_0$. If $z = w$, we have $k - 1$ edges joining w and y_1, which then by inductive hypothesis are joined by a path of edges. If $z \neq w$, project upon $[zy_1]$. w goes to z, and we have $k - 1$ or fewer edges joining z to y_1 and again a path of edges joining them. In either case, by 1.13, we have a path of edges from y_0 to y_1. The points where the edges tick are in $[y_0y_1]$, so the whole path is. A final induction completes the proof.

We shall say that x *splits* a and b if $[ax] \cap [xb] = \{x\}$.

1.16. *The points of a medium M which split two given points a, b, constitute a modular lattice L with $x \vee y = (a, x, y)$ and $x \wedge y = (b, x, y)$. L is a submedium of M, with the medium structure induced by the lattice structure.*

PROOF. Putting $j = (a, x, y)$, $[aj]$ ticks $[jx]$; so by 1.13, j splits a and b. Partially order L by $x \leqslant y$ if $y \in [ax]$. This is plainly reflexive and transitive. If $x \leqslant y \leqslant x$, we have $[ax] = [ay]$; since each z in L is $(b; [az])$, $x = y$. Now note that if $y \in [ax]$, x and y in L, then a, y, x, b determine a path and $x \in [by]$. The converse is true by the same argument; $y \in [ax]$ if and only if $x \in [by]$. We have $j \geqslant x$ and $j \geqslant y$. Also $j \in [xy]$; so if v is a common upper bound of x and y, $[bv]$ contains $[xy]$ and j. So L is a semilattice and, exchanging a and b, a lattice.

Now (x, a, y) splits x and a and is thus in L. Being in $[xy]$, it is $\leqslant j$; being in $[ax] \cap [ay]$, it is $\geqslant j$. So $j = (x, a, y)$. We can now show that for any z in L, $(z \wedge j) \vee m = f$ is (z, x, y). Then by exchanging a and b, the dual formula must also give (z, x, y), and the proof will be complete. Observe that at least, f is $(z \wedge j; [xy])$ since $(z \wedge j) \vee m = (z \wedge j, a, m)$, so that $[(z \wedge j)f]$ ticks $[am] \supset [xy]$. Similarly $[z(z \wedge j)]$ ticks $[bj]$. Then by 1.13, z, $z \wedge j$, f, t determine a path for any $t \in [xy]$. Thus $[zf]$ ticks $[xy]$ and $f = (z, x, y)$.

We sharpen a point from the proof of 1.16.

1.17. *If both x and y split a from b then $(a, x, y) = (x, a, y) = (y, a, x)$, and it is the only point common to $[ax]$, $[ay]$, and $[xy]$.*

PROOF. We saw that the join j is $(a, x, y) = (x, a, y)$, and symmetrically it is (y, a, x). Suppose p belongs to the three indicated intervals. j splits x from $p \in [ay]$, so $j \in [xp]$, $[jp] \subset [xp]$. By 1.7, $(b, j, p) = ((b, x, p), j, p) = (x, j, p) = j$. That is, $[bj]$ ticks $[jp]$. But it contains x and y, thus also $p \in [xy]$ and $[jp]$; so $[jp]$ is a singleton, $p = j$.

1.18. *Given two paths from a to b, a nonsingleton interval in the first path is perspective with at most one interval in the second path.*

PROOF. Having 1.16, this is not novel. We are in a modular lattice with zero, b, and one, a. Given $x_{i-1} \geqslant x_i$, and $a = y_0 \geqslant y_1 \geqslant \cdots \geqslant y_n = b$, given that projection to $[x_{i-1}x_i]$ takes y_{j-1} to x_{i-1} and y_j to x_i, from monotonicity of projection we see that $y_0, \ldots, y_{j-2}$ project to x_{i-1} and $y_{j+1}, \ldots, y_n$ to x_i.

(xvi) 1.18 does not extend to any interval $[xx']$ vis-a-vis a path. In any projective plane, let A be a line and b a point not on A. There are paths of edges determined by A, p, L, b, where p is any point of A and L the line joining p to b. If x is any point not on A or L, and x' a line through x not through p or b, then $[xx']$ is perspective with all three intervals of the path.

1.19. THEOREM. *Any two paths of edges from a to b have the same number of terms, and parallelism is a bijective relation between them.*

PROOF. Since we are in the modular lattice of 1.16, this is substantially a known result implicit in work of Zassenhaus [**10**] and Birkhoff [**2**]. To render it explicit, let the vertices be $a = x_0, x_1, \ldots, x_m = b$, and $a = y_0, y_1, \ldots, y_n = b$. Consider the projections $\xi_{ij} = (y_j, x_{i-1}, x_i)$. For each i, $\xi_{i0} = x_{i-1}$ and $\xi_{in} = x_i$. So for some j, $\xi_{ij-1} \neq \xi_{ij}$. By 1.12, $[y_{j-1}y_j]$ is then parallel to $[x_{i-1}x_i]$. By 1.17, ξ_{ik} is x_{i-1} for $k \leqslant j-1$ and x_i for $k \geqslant j$. This shows that parallelism is a function in one direction. But the same is true in the other direction, and we are done.

In an edge-connected medium we define the distance $\rho(a, b)$ as the number of edges in a path from a to b. We have

1.20. $\rho(a, c) \leqslant \rho(a, b) + \rho(b, c)$, *with equality if and only if* b *splits* a *and* c.

PROOF. In case $\rho(b, c) = 1$, consider (a, b, c). It is b precisely when b splits, and then there is a path of edges from a via b to c. When it is not b, $\rho(a, b) = 1 + \rho(a, (a, b, c)) \geqslant \rho(a, c)$. So the assertion holds in this case, and a straightforward induction gives the general case.

COROLLARY. *Every string of* $\rho(a, b)$ *edges joining* a *to* b *is a path.*

PROOF. All the vertices split a and b, so a trivial induction establishes this.

(xvii) $[ab] = [cd]$ does not imply $\rho(a, b) = \rho(c, d)$. Indeed, $[ab] = [ac]$ and a edge-connected to b do not imply a edge-connected to c. For take points $a, b, c, 0, 1, \ldots,$ and let the nontrivial ideals be (1) those of the totally ordered set $a < 0 < 1 < \ldots < c$, and (2) those of the totally ordered set $b < 0 < 1 < \ldots < c$. If finitely many integers are used, this proves the first assertion; if infinitely many, the second.

Now we can easily characterize Čebyšev sets in edge-connected media. Note

1.21. *A Čebyšev set S contains every path of edges joining two points of S.*

PROOF. The corollary to 1.6 gets the splitting points, so it remains only to note that a Čebyšev set containing two points of an edge E cannot omit another point of E.

1.22. THEOREM. *The Čebyšev sets of an edge-connected medium are precisely the nonempty sets that contain all paths of edges between their points.*

PROOF. For any nonempty set S in an edge-connected medium, and any point x, there is a point m of S at minimum distance from x. S is Čebyšev if always $(s, m, x) = m$, for s in S. So suppose $(s, m, x) = f \neq m$. Assuming S contains paths of edges, it contains f which splits s and m. In case $[mf]$ is an edge, it is in S and (x, m, f) must be the minimum-distance point m; so we have a contradiction of $f \in [xm]$. Otherwise split m, f, with c at distance 1 from f. c is in S, and as before $(x, m, c) = m$. But this makes $c \notin [xm]$, though $c \in [mf]$ and $f \in [xm]$, a contradiction.

Edge-connected media all of whose Čebyšev sets are ideals are called *reduced.* From 1.9 and 1.10, clearly, every edge-connected medium determines a reduced medium with the same Čebyšev sets and edges.

Note (xviii) by 1.12, projection upon a Čebyšev set is distance-decreasing.

2. Tautness and rectangles.

2.1. *In a submedium of a modular lattice, if b is between a and c then it is between a and (c, a, b).*

PROOF. $a \wedge (c, a, b) = a \wedge (c \vee (a \wedge b)) \wedge (a \vee b) = a \wedge (c \vee (a \wedge b)) = (a \wedge c) \vee (a \wedge b) \leqslant b$; dually, $a \vee (c, a, b) \geqslant b$.

This conclusion can be written as an identity, since the most general b in $[ac]$ is precisely (x, a, c). We call the property *tautness*.

$$((x, y, z), y, (z, (x, y, z), y)) = (x, y, z). \tag{T}$$

2.2. *In a taut medium, if b is in $[ac]$ then $a' = (a, b, c)$ and $c' = (c, b, a)$ satisfy $[a'c'] = [a'b] = [bc']$.*

PROOF. Since $b \in [a'c]$, b is between a' and $c'' = (c, a', b)$. But by Theorem 1.8, c'' is $(c', a', b) \in [a'c']$, so $b \in [a'c']$.

Similarly $a' \in [bc]$, so $a' \in [bc''] \subset [bc']$. Symmetrically $c' \in [a'b]$.

COROLLARY. *A free taut medium on three generators has six elements and all its homomorphic images are embeddable in modular lattices.*

PROOF. On generators a, b, c, we get $a' = (a, b, c)$ and symmetrically b', c'. a' does not split b and c because splitting is expressible by an equation and there is a taut medium (an edge $\{a, b, c\}$) in which it fails. So there are at least these six elements and, by 2.2, $[a'b'] = [a'c'] = [b'c']$. But these six elements form a submedium. For consider the three sets $\{a, a'\}$, $\{b, b'\}$, $\{c, c'\}$. If $\{x, y, z\}$ meets all three then (x, y, z) is x'. If x, y, z are not all different we know (x, y, z). The remaining cases permute to (a, a', b) (a', a, b), and (b, a, a'), all of which are a' since a' splits a and b.

There exist three elements of a modular lattice generating a six-element medium M. In a vector space with basis $\{p, q, r, s, t\}$, span a by $\{p, q, s\}$, b by $\{p, r, t\}$, c by $\{q, r, s + t\}$, and check. Since M is taut, it is free taut. It is routine to check that it has just seven nonisomorphic proper quotients, six embeddable in M and the last arising when a', b', and c' are identified. All go in modular lattices.

I do not know whether a taut medium (in a modular lattice) generated by four elements must be finite. Example (x) is an infinite medium generated by three elements.

A couple of observations about taut media.

(xix) If $b \in [ac]$ and $[ab] \subset [a^*b^*]$, then $b \in [a(c, a^*, b^*)]$. Using Theorem 1.8, this is simple routine.

(xx) If $b \in [ac]$ and S is a Čebyšev set containing a, then $(b; S)$ is between a and $(c; S)$. For $[ac]$, containing b and $a \in S$, contains $b' = (b; S)$. By tautness b' is between a and $c^* = (c, a, b') = ((c; S), a, b')$, and $[ac^*] \subset [a(c; S)]$.

2.3. THEOREM. *In a taut medium,*

(a) *perspective intervals are parallel*;

(b) *parallel intervals are isomorphic.*

PROOF. (a) Suppose $[ab]$ and $[xy]$ are perspective as named and $p \in [xy]$ is not $((p, a, b),\ x, y) = p^*$. Put $c = (p, a, b) = (p^*, a, b)$ (since $[pc]$ contains p^* and ticks $[ab]$). Note, for all q in $[pp^*] \subset [pc]$, $(q, a, b) = c$. Now $[a(a, p, p^*)]$ contains (1) c and (2) x. By (1) and $(c, p, p^*) = p^*$, $(a, p, p^*) = p^*$. By (2), then, $(x, p, p^*) = p^*$. Similarly $(y, p, p^*) = p^*$.

In particular, $[xp]$ and $[py]$ do not tick since p^* is in both. That is, any r which does split x and y is $((r, a, b),\ x, y)$. This applies to $x' = (x, p, y)$ and to $y' = (y, p, x) = (y, p, x')$. We have p and p^* in $[x'y']$ which is perspective as named with $[a'b']$ for suitable a', b' in $[ab]$. Moreover, $[x'p] = [y'p] = [x'y']$ by 2.2.

Between a' and p are x', hence y' and b'; so by (xix), b' is between a' and $(p, a, b) = c$. Between a' and y' are x', hence p; by (xix), $c \in [a'b']$. Similarly $a' \in [b'c] = [a'c] = [a'b']$. But $(b', p, p^*) = (y', p, p^*) = p^*$, while $[b'p^*]$ contains c, a', x' and therefore p, a contradiction.

(b) Intervals I, J being parallel, we have a bijection $f\colon I \to J$ taking each x to $(x; J)$, whose inverse has the same form. By Theorem 1.3, f is an isomorphism if f and f^{-1} preserve betweenness. It suffices to show that f^{-1} preserves betweenness. If $f(a) \in [f(b)f(c)]$ in J, then since $[bf(c)]$ contains $(b; J) = f(b)$, it contains $f(a)$ and $(f(a); I) = a$. By (xviii), a is between b and $(f(c); I) = c$.

Now the Zassenhaus-Birkhoff theorem. We have already used (in 1.15) the obvious notion of a refinement of a path. To reduce fuss let us identify any two paths that have the same nonsingleton terms, i.e. that refine each other. Call two paths *piecewise parallel* if there is a bijection between their nonsingleton terms which is contained in the parallelism relation. (By 1.17, if piecewise parallel paths have the same ends, the bijection is unique.)

2.4. THEOREM. *In a taut medium, two paths from a to b have coarsest piecewise parallel refinements.*

PROOF. The paths, $\{[x_{i-1}x_i]\}$ and $\{[y_{j-1}y_j]\}$, are given by chains in the lattice L of splitting points, $a = x_0 \geqslant x_1 \geqslant \ldots \geqslant x_m = b$ and $a = y_0 \geqslant y_1 \geqslant \ldots \geqslant y_n = b$. All the elements $\xi_{ij} = (y_j, x_{i-1}, x_i)$ form a chain; for fixed i, they form a chain $\xi_{i0} = x_{i-1}$ to $\xi_{in} = x_i$, monotonically since projection is a lattice polynomial. This chain determines a path whose nonsingleton intervals I_{ij} all have the form $[\xi_{ij-1}\xi_{ij}]$. Similarly for $\eta_{ji} = (x_i, y_{j-1}, y_j)$. The intervals I_{ij} and $[\eta_{ji-1}\eta_{ji}]$ are perspective, hence parallel. Finally, for intervals with ends in L, a subinterval of $[x_{i-1}x_i]$ which is perspective with a subinterval of $[y_{j-1}y_j]$ must be contained in I_{ij} since lattice operations are monotonic.

One inclines to describe the Zassenhaus construction in 2.4 as "projecting the paths upon each other". I do not know whether it is that or not. Prima facie we projected the ends and used whatever intervals they happen to mark out. The projections of the given intervals contain these (since there are subintervals projecting bijectively). In general, projections of intervals are not intervals; it is convenient to defer the proof to 2.8. Now, a little more on projections and parallels.

Call two Čebyšev ideals I, J *parallel* if projection upon J maps I bijectively. Just as for intervals (xiii), this is a symmetric relation. From the proof of 2.3(b), (xxi) parallel Čebyšev ideals in a taut medium are isomorphic.

Later (2.12) we shall, rather surprisingly, have a use for parallelism of sets which are perhaps not Čebyšev. Define a *cross* of two sets S, T, as a minimal interval ticking both of them. Evidently an interval ticking S and T, at s and t respectively, contains a cross $[st]$. Call two sets *parallel* if they are covered by their crosses. For Čebyšev ideals this agrees with the previous definition.

2.5. *In a taut medium, if I and J are Čebyšev ideals, the projections $(I; J)$ and $(J; I)$ are parallel Čebyšev ideals.*

Proof. For any $x^0 \in I$ and $y \in (x^0; J)$, let $x = (y; I)$. Since $x \in [x^0y]$ which ticks J, $y = (x; J)$. Now consider $y_2 \in [y_1y_3]$, y_1 and y_3 in $(I; J)$. $[y_1y_3]$ is perspective to $[x_1x_3]$, therefore parallel; so for $x^* = (y_2, x_1, x_3) \in I$ we have $(x^*, y_1, y_3) = y_2$. Also $[x_1y_1]$ and $[x^*y_2]$ are perspective and hence parallel. Now $(x^*; J) \in [x^*y_2]$. If it were different from y_2, parallelism would take it to $z \neq y_1$ in $[x_1y_1]$. But z is between $(x^*; J)$ and y_1, hence in J; since $[x_1y_1]$ ticks J at y_1, this is impossible. Thus $y_2 = (x^*; J)$ and $(I; J)$ is an ideal.

To show that $(J; I)$ is Čebyšev it suffices to show that it is relatively Čebyšev in I (1.7). For x^0 in I, $[x^0(x^0; J)]$ contains $((x^0; J); I)$ as we noted, but it can contain no other point of $(J; I)$ since it ticks J. So $(J; I)$ is Čebyšev. Similarly $(I; J)$ is Čebyšev, and we have these sets covered by their crosses.

Corollary. *In a taut medium, the projection of an ideal in a Čebyšev ideal is an ideal.*

Proof. Any two of its points lie in the projection of an interval.

Recall (not for use; only bringing like remarks together) that we noted after 1.6 that each Čebyšev set is a submedium. Evidently also, the union of two ideals is a submedium. Given the medium structure of the ideals I, J, what about the structure of $I \cup J$? This is a complex question, and we examine only the simplest case. Given two media M_1, M_2, and a medium I embedded in M_1 by f_1 and embedded in M_2 by f_2, if each $f_j(I)$ is a Čebyšev ideal in M_j then we can define the *amalgam* of M_1 and M_2 along I as follows. Its points are those of I, of $M_1 - I$, and of $M_2 - I$. To simplify notation, identify I with $f_1(I)$ and with $f_2(I)$, so that the amalgam is $M_1 \cup M_2$. The interval $[xy]$ is the interval in M_j if $M_j \supset \{x, y\}$, but for $x \in M_1, y \in M_2$, $[xy] = [x(y; I)] \cup [(x; I)y]$. We can say explicitly what (x, y, z) is (viz. $((x; I), y, z)$ if $M_j \supset \{y, z\}$ and $(x, y, (z; I))$ if $M_j \supset \{x, y\}$); but the verification that we have a medium is easier via Theorem 1.3. So define an ideal of $M_1 \cup M_2$ as a set which contains the interval between any two of its points.

2.6. *The amalgam of two media along a common Čebyšev ideal is a medium containing the given media as Čebyšev ideals. It is taut if the given media are.*

The proof is routine.

(xxii) Note if you wish, these amalgams are not pushouts. In fact they are far from it. Apart from the possibility of other structures on $M_1 \cup M_2$, there are in general media generated by M_1 and M_2 in which M_1 and M_2 are not ideals; and the pushout is universal.

2.7. *A taut medium which is the union of two Čebyšev ideals I, J, that intersect, is the amalgam of I and J along $I \cap J$.*

PROOF. Since the intervals determine the structure, we need only inquire when $x \in [yz]$. If y, z are both in I or both in J, it is obvious. By notational symmetry it remains only to consider x, y in I, z in J. By (xix), $x \in [yz]$ implies $x \in [y(z; I)]$, and the converse is trivial. Now actually by the natural proof of 2.6, this is the condition for $x \in [yz]$ in the amalgam. Since the reader was left to construct a proof, here is the needed part. $[yz]$ in the amalgam contains $[y(z; I)]$, and also $[(y; J)z]$ which can contain x if $x \in I \cap J$. But if it does, then by (xix) in J $[(y; J)(z; I)]$ also contains x; hence so does $[y(z; I)]$.

(xxiii) There is a converse. If every submedium of M satisfies 2.7, then M is taut. Just consider the general union of two ideals $[ab]$ and $[c(c, a, b)]$, for $b \in [ac]$.

(xxiv) A Čebyšev ideal in an interval need not be an interval. In a vector space with basis $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ let M consist of the following 18 elements: the 8 subspaces spanned by subsets of $\{x_1, x_2, x_3\}$, the 8 spanned by subsets of $\{y_1, y_2, y_3\}$ (0 being in both), and the lines spanned by $x_i + y_i$. Evidently the first 15 elements form a submedium A, and the last three with 0 form a submedium B. Observe that the lattice interval spanned by any two elements of M either has infimum 0 or has supremum in A, and it is clear that M is a medium (embedded in a modular lattice). M is an interval $[pq]$, p spanned by x_1, x_2, x_3 and q by y_1, y_2, y_3. But B is a Čebyšev ideal, everything in A having foot 0 in B.

2.8. *The projection of an interval in an interval, in a taut medium, need not be an interval.*

PROOF. Amalgamate two copies of M in (xxiv) along B.

2.9. *In a taut medium, if $(x, y, z) = (y, z, x)$ then it is the only point of $[xy] \cap [xz] \cap [yz]$.*

PROOF. Suppose p belongs to all three intervals. If m is the median (x, y, z), m splits x from $p \in [yz]$ and $m \in [px]$. From $m = (y, z, x)$, $[ym]$ ticks $[xz] \supset [px]$, so $m = (y, p, x)$. By tautness and $p \in [xy]$, $p \in [xm]$. But we had m splitting, $[xm]$ ticks $[mp]$, its subinterval which is therefore a singleton; $p = m$.

(xxv) In general media 2.9 need not hold even if also $(z, x, y) = (x, y, z)$, though 1.17 gives a further condition which does suffice. Simply take x, y, z, their median (in any order) m, and a fifth point p, to belong to $[xy]$, $[xz]$, $[yz]$, and with $[pt] = [mt] \cup \{p\}$ for $t \neq p$. It is easy to check the conditions of 1.3 for the ten nonsingleton intervals.

2.10. THEOREM. *In a taut edge-connected medium, if $[cd]$ is a subinterval of $[ab]$, then $\rho(c, d) \leqslant \rho(a, b)$, with equality only when $[cd] = [ab]$.*

PROOF. First the case $d = a$, by induction on $n = \rho(a, b)$. If both statements hold for $n' < n$, and $[ac]$ is a proper subinterval, consider $b' = (b, a, c)$. It splits a and b

and is not b, so $\rho(a, b') < n$. By tautness $c \in [ab']$, so $\rho(a, c) \leqslant \rho(a, b') < n$. Now for any c' in $[ab]$, it cannot be that $\rho(a, c') > n$; for some c would split a and c' at distance n from a, though $[ac']$ is a proper subinterval of $[ab]$.

For general $d \in [ab]$ and any c, we prove $\rho(d, (d, a, c)) \leqslant \rho(b, c)$. It suffices to treat the case $\rho(b, c) = 1$; the general case then goes as follows. Split b and c by p at distance 1 from b. By inductive hypothesis,

$$\rho((d, a, p), ((d, a, p), a, c)) \leqslant \rho(p, c) = \rho(b, c) - 1.$$

Then $\rho(d, (d, a, p)) \leqslant 1$ will put d within $\rho(b, c)$ of some points of $[ac]$, and thus of (d, a, c). So, in the case $\rho(b, c) = 1$, let $b^* = (b, a, d)$, $c^* = (c, a, d)$. By tautness, $d \in [ab^*]$ and $[ab^*] = [ad]$. $(d, a, c) = d' \in [ac]$, so it is in $[a(c, a, d')] \subset [ac^*]$; similarly $c^* \in [ad']$: Now d' splits a and d. Further, $\rho(b^*, c^*) \leqslant 1$; therefore

$$\rho(d, d') = \rho(a, d) - \rho(a, d') = \rho(a, b^*) - \rho(a, c^*) \leqslant 1.$$

Finally, the theorem by induction on $\rho(a, b)$. $C = (b, a, c)$ splits a and b; if it is $b \in [ac]$ we have $[ab] = [ac]$ and two applications of the first case suffices. Otherwise $[aC]$ is shorter, and $c \in [aC]$, so $\rho(c, (d, a, C)) \leqslant \rho(a, C)$. Equality implies $a \in [c(d, a, C)] \subset [cd]$. Also $\rho(d, (d, a, C)) \leqslant \rho(b, C)$; so $\rho(c, d) \leqslant \rho(a, b)$, and equality implies $a \in [cd]$. But then symmetrically, equality implies $b \in [cd]$.

COROLLARY. *Taut edge-connected media are artinian.*

One might speculate that Theorem 2.10 would imply a Jordan-Dedekind chain condition (1) for the partially ordered set of subintervals of a finite-length taut interval. No; in M of (xxiii), a maximal subinterval of B has length 2 and is contained in no subinterval of length 3. In M, though, the lattice of ideals is Jordan-Dedekind. So consider (2) seven subspaces of a projective plane, the empty subspace 0, points making a triangle abc, and further points (ab), (ac), (bc) on its sides, not collinear with each other. (If there are four points on a line, this is possible.) It is easy to see, indeed we will use again:

(xxvi) In a projective space, 0 and any set of points form a submedium.

In this one, a maximal chain of ideals up from $\{0, (ab)\}$ can take two steps, adding a (hence b), then c (hence everything); or it can take three steps via (ac) and (bc).

(xxvii) For an edge-connected nonartinian medium, one can take the natural numbers, defining an ideal as a set which, if it contains two elements, contains 0, and if it contains 1 and n, contains $n + 1$. There is no trouble in checking that this gives a medium, and the intervals $[1n]$ for $n = 2, 3, \ldots,$ decrease, though any two elements are joined by two edges.

(xxviii) Even in a submedium of a modular lattice, two chains of edges across an interval $[ab] = [cd]$ need have no more in common than the theorem says. In a vector space with basis $\{w, x, y, z\}$, the subspaces spanned by $\{w, x\}$, $\{y, z\}$, $\{w, y\}$, $\{w + y, x + z\}$, and $\{w + y, w + x + z\}$ form a medium. First to third to second is a path of two-element edges across; similarly first to third to fifth, but the latter edge has three elements.

In a modular lattice, if $[ab] = [cd]$, then paths from a to b and from c to d both have refinements piecewise parallel to paths from $a \wedge b$ to $a \vee b$. (No difficulty. One cannot make them piecewise parallel to each other, in any nondistributive lattice.) I am indebted to Stephen Schanuel for pointing out a conjectural explanation of Theorem 2.10. If we had a natural, edge-connected, containing modular lattice, the equinumerous edges would become complexes of "imaginary factors", the same for both paths; and it is easy to guess that a natural construction would break up all edges into the same number of imaginary factors. However, modular lattices cannot hope to do this in full generality, because they do not contain all these media.

2.11. *In a submedium of a modular lattice, if* $[ab]$ *and* $[xy]$ *are perspective as named, then any two ideals* I, *meeting both* $[ab]$ *and* $[xy]$, *and* J, *meeting both* $[ax]$ *and* $[by]$, *meet each other.*

PROOF. We may as well take I and J to be intervals since they contain intervals satisfying the same conditions. If I has infimum p and supremum q in the lattice, the conditions are $p \leqslant (a \vee b) \wedge (x \vee y)$, $q \geqslant (a \wedge b) \vee (x \wedge y)$. Similarly J is given by $[r, s]$ with $r \leqslant (a \vee x) \wedge (b \vee y)$ and $s \geqslant (a \wedge x) \vee (b \wedge y)$. For them to meet means $p \vee r \leqslant q \wedge s$. We have $p \leqslant q$ and $r \leqslant s$. It remains to show $p \leqslant s$ and–symmetrically–$r \leqslant q$. So it suffices to prove

$$(a \vee b) \wedge (x \vee y) \leqslant (a \wedge x) \vee (b \wedge y).$$

Inspection shows that the reverse inequality holds; $a \vee b$ exceeds $j = (a \wedge x) \vee (b \wedge y)$, so does $x \vee y$, and thus also their infimum m. By modularity and comparability of j and m, we need but show that they have the same supremum and infimum with $a \wedge b$. Now $j \vee (a \wedge b) = (a \wedge x) \vee (b \wedge y) \vee (a \wedge b)$ is $\geqslant b = (b \wedge y) \vee (a \wedge b)$ (since $[by]$ ticks $[ab]$), and similarly it is $\geqslant a = (a \wedge x) \vee (a \wedge b)$. But $a \vee b \geqslant m$ too, so the suprema are the same. For infima we want

$$(a \vee b) \wedge (x \vee y) \wedge a \wedge b \leqslant [(a \wedge x) \vee (b \wedge y)] \wedge a \wedge b,$$

which reduces to

$$t = (x \vee y) \wedge a \wedge b \leqslant (a \wedge x) \vee (b \wedge y).$$

Now $t \leqslant a \wedge b$ by inspection. Also $t \leqslant (x \vee y) \wedge (a \vee x) = x$; and $t \leqslant (x \vee y) \wedge (b \vee y) = y$. Hence the desired relation.

We call this property the *axiom of Pasch*. It does not imply tautness. (If there are no nontrivial rectangles, the Pasch axiom holds but tautness need not.) But it seems more interesting for taut media; so we define a *Paschian* medium as a taut medium satisfying the axiom of Pasch.

The axiom of Pasch is a medium identity. For, first, the general rectangle $abyx$ is represented by $a = (p, q, r)$, $y = (p, q, s)$, $b = (p, (p, q, r), (p, q, s))$, $x = (q, (p, q, r), (p, q, s))$. (Thus a and y split p and q, and $b = a \vee y$, $x = a \wedge y$, in the lattice of splitting points.) Then, the general point e of $[ab]$ is (t, a, b); general $f \in [ax]$ is (u, a, x). The intervals $[e(e, x, y)]$ and $[f(f, b, y)]$ are supposed to meet, and if they meet at all they contain $(e, f, (f, b, y)) = g$. Certainly g is in the

f-interval, so the axiom is

$$(g, e, (e, x, y)) = g, \tag{P}$$

which the patient reader may put in terms of p, q, r, s, t, u.

Six variables seem to be necessary to express Pasch's axiom. It is interesting to note that it is equivalent to a four-variable relation in the lattice constructed in Theorem 1.4: namely, the identity $j = m$ from the proof of 2.11. This is taking a, b, x, y to be points of the medium; probably the lattice will not satisfy such an identity throughout. (It is not clear how to state it, since the two lattice formulas for median do not agree.)

I do not know if submedia of modular lattices satisfy further conditions like (P). The conditions we have suffice for the following. Note, two crosses of a pair of ideals are perspective, so in taut media, parallel.

2.12. *The union of the parallelisms between crosses of two parallel Čebyšev ideals in a Paschian medium is an equivalence relation, and the equivalence classes are parallel ideals.*

PROOF. First consider parallel intervals $[ab]$, $[xy]$. Parallelism means that the crosses $[cz]$ correspond bijectively to the points c of $[ab]$, and also to the points z of $[xy]$. Similarly for the crosses $[fm]$ of $[ax]$ and $[by]$. Now $[cz]$ contains (by parallelism) exactly one point p_1 satisfying $(p_1, a, x) = f$, and one p_2 satisfying $(p_2, b, y) = m$. Since points of $[fm]$ satisfy both equations, $[cz]$ has at most one of those, and by Pasch's axiom it has one. These ticking relations show that the collection of images of $p \in [cz]$ under the parallelisms in question is a cross of $[ax]$ and $[by]$, thus an ideal, and that the relation is an equivalence.

For general parallel Čebyšev ideals I, J, consider three crosses $[ax]$, $[by]$, $[cz]$, and their respective points, say $p \mapsto q \mapsto r$. There is a point d, for instance (a, b, c), in $[ab] \cap [ac] \cap [bc] \subset I$. The cross $[dw]$ contains a unique point s corresponding by parallelism to q. By the preceding paragraph, appropriate parallelisms take p to s, s to r, and (therefore) p to r. So these relations are equivalences. Again by the preceding paragraph, each equivalence class contains with any p and q the interval $[pq]$.

We turn back to what are perhaps the simplest media after edges (or after bouquets of edges): 2-*cells*, defined as media containing four (edge-connected) points a, b, c, d, whose ρ-distances are 1 except for $\rho(a, c) = \rho(b, d) = 2$, and such that no proper ideal contains $\{a, b, c, d\}$.

2.13. THEOREM. *2-cells are taut and reduced. The set $\mathfrak{E}$ of edges of a 2-cell has the properties* (0) *there exist two disjoint edges, and two points not in an edge,* (1) *no two edges have two common points,* (2) *if three edges intersect pairwise they are concurrent, and* (3) *the edges intersecting a given edge cover the points. Any set $\mathfrak{E}$ of nonempty subsets of a set M satisfying these conditions is the set of edges of a unique 2-cell on M.*

PROOF. (0), (1), and (2) are evident. Next, call an ordered quadruple (a, b, c, d) as in the definition a *frame* of the 2-cell M. Observe that $[ac]$ contains b and d, so

$[ac] = M$; similarly $[bd] = M$. In the lattice L of points splitting a and c, each maximal chain has three elements (counting a and c); for this is true of chains containing b, since $[ab]$ and $[bc]$ are edges, and of other chains by modularity of L. We show next that every point x belongs to a frame. This is certainly true if (α) x belongs to an edge $[ab]$ of a frame; for the perspective edges $[ab]$, $[dc]$ are parallel, x belongs to a cross $[xy]$, (a, x, y, d) gives a rectangle of edges, and no ideal contains $\{a, x, y, d\}$ without containing $b \in [ax]$ and everything. Now consider any other x. By (α), (x, a, b) belongs to a frame; so we may as well call such a frame (A, B, C, D) and have $(x, A, B) = A$. $[xA] \neq M$ since it ticks $[AB]$. It also ticks $[CF]$ where $F = (C, x, A)$. (So $[CF] \neq M$, $F \neq A$.) Since F splits A and C, $[AF]$ and $[CF]$ are edges. F is not in $[BC]$ since $[AF]$ ticks $[AB]$; so (A, B, C, F) gives a rectangle of edges, and since $[AC] = M$ it is a frame. If $x \in [AF]$, it is in a frame by (α). Suppose $x \notin [AF]$, so that $[AF]$ just ticks $[xG]$ where $G = (x, A, F)$. Everything in $[Ax]$, in particular (B, x, G), projects to A in $[AB]$. So $[B(B, x, G)]$ contains A, and with it $(A, x, G) = G$. G is not A since the edge $[AF] \subset [Ax]$ ticks $[Gx]$. So $F \in [AG] \subset [B(B, x, G)]$. But $[BF] = M$. Hence $x = G \in [AF]$, a contradiction; every point is in a frame.

Any edge has the form $[ax]$ with a in a frame (a, b, c, d), and as above it is $[af]$ where (a, b, c, f) is a frame. Again consider the above argument; we took any such edge $[ab]$ and found any x to be within an edge of (x, a, b). So (3) holds.

It follows that an interval $[ax]$ which is not an edge is all of M; so M is taut. (Two of $[a(c, a, b)] \subset [ab] \subset [ac]$ have to coincide since (c, a, b) cannot be a when $b \in [ac]$.)

The diameter of M is 2. A Čebyšev set of diameter 0 or 1 is evidently an ideal. By Theorem 2.10, every nondegenerate rectangle is a frame. Then another inspection of the main proof shows that arbitrary x turns out to be obtainable from $\{a, c\}$ by closing under taking joining paths of edges. So M is reduced.

Given a set M and set of subsets $\mathfrak{E}$, "edges", satisfying (0)–(3), not all edges are singletons by (0) and (3). If one edge were a singleton, so would all edges disjoint from it be singletons, by (1), (2), and (3). Then it is easy to check that M would be an edge, violating (0); so all edges are nonsingleton and proper subsets. Adjoining singletons and M and the empty set to $\mathfrak{E}$, we get a family $\mathcal{G}$ satisfying the conditions of 1.3. So M is made into a medium, $\mathfrak{E}$ is the set of edges, and there is a rectangle (a, b, c, d) by (0). By (3), any point x is within an edge of some $y \in [ab]$ and some $z \in [cd]$. If those two edges coincide $x \in [ac]$. If they differ, $\rho(y, z)$ is not 1, and still $x \in [yz] \subset [ac]$. So (a, b, c, d) is a frame of M: a 2-cell. Evidently $\mathfrak{E}$ determines the medium structure.

Corollary. *The edges of a 2-cell M are the points of a 2-cell M^*, whose edges are the stars of points of M. M^{**} is isomorphic with M.*

Proof. Each of (0)–(3) for M implies the same for the proposed structure on M^*; and evidently M^{**} is naturally isomorphic with M.

We take up n-cells to show that the obvious definition works, giving frames, tautness, reducedness and roughly the same abundance of frames we found in

2-cells. Inductively, a $(k + 1)$-*cell* is a medium containing two perspective k-cell ideals I, J, whose crosses are edges and whose union is in no proper ideal. "Perspective" is meaningful since, by induction, cells are intervals. It is useful because:

2.14. *Perspective reduced intervals* $[x_1 y_1]$, $[x_2 y_2]$, *crossed by edges* $[x_1 x_2]$, $[y_1 y_2]$, *are parallel and isomorphic.*

Proof. The beginning of the proof of 2.3 does not use tautness and shows that every splitting point p lies in a cross $[pq]$. q is splitting since the function ρ is preserved and determines splitting. Then a path of edges along $[x_1 y_1] = I_1$ projects bijectively to a path of edges along $[x_2 y_2] = I_2$. More, it is covered by crosses. For consider r in an edge $[pt]$ in a path from x_1 to y_1. p is in a cross $[pq]$, an edge because it is perspective with an edge. r is in a cross $[rs]$ of $[pt]$ and its projection in I_2, and $[rs]$ is an edge. Then I_1 ticks $[rs]$ because it does not contain it. Similarly I_2 ticks $[rs]$, which is thus a cross. Evidently this continues; the subset of I_1 covered by crosses of I_1 and I_2 is closed under taking paths of edges. Thus the intervals are parallel. The parallelism preserves ρ and is thus isomorphic.

Back to cells: a *frame* of a $(k + 1)$-cell is given by a suitable ordered pair of k-cells I, J (as in the definition) and a frame of I. Observe that, expanding this, the frame is a combinatorial cube with the same additional information that an ordering of coordinates would give (I being the face "$x_{k+1} = 0$").

2.15. *n-cells are taut and reduced, and any two points are vertices of a frame.*

Proof. First go back and observe that any two points x, c, of a 2-cell are vertices of a frame. We stated the result for one point, and also for an edge, thus for two points at distance 1. For $\rho(x, c) = 2$, put c in a frame (a, b, c, d) (in the new convention, $(([ab], [cd]), (a, b)))$ and look at the argument (α). We adduced the frame (a, x, y, d), but could as well use (x, b, c, y). Then $C = c$ and it is in the final frame (A, B, C, F). But we can replace A and B with x and (x, B, C).

Now assuming 2.15 for k-cells, consider a $(k + 1)$-cell between parallel plates I, J, with a frame for $I = [ab]$. We want to get one point x in a frame. By induction we have it if $x \in I$, or if x is anywhere else in the k-skeleton of the given combinatorial frame. So as before, we have $(x; I) = A$ in a frame, say $I = [AB]$ perspective with $J = [YZ]$. We get F splitting A and Z; take E splitting A and F at distance 1 from A. In the lattice of points splitting A and Z, $B \vee E = A$ since it is in I (and $(E; I) = (x; I) = A$). $B \wedge E = Z$ since B is a minimal element of the lattice in I. So $[EZ]$ is perspective to I and is another k-cell. Now as before, $x \notin [AE]$ is impossible. Therefore x is in a frame. Therefore in working with two points, we may assume they were x and a; and when we put $(x; I) = A$ in a frame of I, by inductive hypothesis we may take that frame to have a also as a vertex. Now both tautness and reducedness are quite easy.

n-cells are, roughly, "saturated" intervals of length n. A precise statement:

2.16. *If a medium M contains an n-cell C whose distance function ρ is preserved by $C \subset M$, then the smallest ideal of M that contains C is an n-cell.*

The proof is a trivial induction. Another way of putting it is that if $M = [ay]$ where $\rho(a, y) = n$ and the lattice of points splitting a and y contains $\mathbf{2}^n$ (or equivalently, it is complemented), then M is an n-cell.

(xxix) An n-dimensional projective geometry, as a medium, is an $(n + 1)$-cell. The 2-cell faces are of course just (certain) bipartite graphs.

Returning to 2-cells, the projective plane-like description by (0)–(3) in 2.13 suggests defining a *partial* 2-*cell* as a pair of sets M, E, with a relation "on" between M and E such that (1) no two elements of E are on two elements of M, and (2) no three elements of E are pairwise on three distinct elements of M. Call it *nondegenerate* if (0) holds (some two elements of M on no common $e \in E$, and dually).

In this paper, no use will be made of partial 2-cells except to show that 2-cells exist which are non-Paschian–which we will soon illustrate with natural examples–and that almost no laws govern which crosses of rectangles meet. We define the *free closure* of a nondegenerate partial 2-cell on M and E as follows. Put $M_0 = M$, $E_0 = E$. Having a partial 2-cell on M_i and E_i, form M_{i+1} by adjoining to M_i a symbol $(x; e)$ for each x in M_i and e in E_i not on x. For E_{i+1}, adjoin to E_i symbols $[x(x; e)]$ for the same pairs x, e. Extend "on" by defining $(x; e)$ to be on e and $[x(x; e)]$ and defining x to be on $[x(x; e)]$. Completing the induction, the free closure on M^-, E^- is defined by $M^- = \bigcup M_i$, $E^- = \bigcup E_i$, and the relation "on" is likewise the union.

2.17. *The free closure of a nondegenerate partial* 2-*cell is a nondegenerate partial* 2-*cell satisfying* (3) *of* 2.13, *and thus it determines a* 2-*cell. Edges not on any common point remain so in the free closure.*

The proof is simple routine and is omitted.

The simplest types of 2-cell are (1) products of two edges $E \times F = M$, and (2) the duals M^* of those, which are called *bipartite graphs*.

2.18. *If some edge of a* 2-*cell M has only two points, then M is either a product of two edges or a bipartite graph. Hence the same is true if some point is on only two edges. If some point is on three or more edges, then all edges are transparallel*; *if some edge has three or more points, then all points are on the same number of edges.*

PROOF. Suppose first that every edge has just two points. Then if $\{a, b\}$ is one edge, (x, a, b) is always a or b. If A consists of a and the points $x \neq b$ for which (x, a, b) is b, then $\{x, b\}$ is an edge for each x in A. Similarly $\{y, a\}$ is an edge for each y not in A. For x in A and y not, (x, y, a) cannot be a, so it is y; $\{x, y\}$ is an edge. We have all the edges of a bipartite graph, and no other doubleton can be an edge.

We prove next that if some point p is on three edges E, F, G, all edges are transparallel. An edge not on p is disjoint from at least two of E, F, G, so parallel to them. Then two edges not containing p have a common parallel; so all those are transparallel, and an edge on p is disjoint from some edge not on p.

Now suppose some edge E has two points and another edge F has more than two points. Then no point of the 2-cell M is on more than two edges. Hence M^* is a bipartite graph, and M a product $E \times F$. The rest of the proposition is immediate from duality.

The only other 2-cells known to be embeddable in modular lattices arise in projective geometries PG(3, k) as follows. If k is commutative (and not otherwise), there is an involutory selfduality of PG(3, k) taking each point p to a plane on p. (See pp. 106–109 of [1].) In homogeneous coordinates, the matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

gives such a duality δ (and all others are conjugate to this one [1]). It is called a *null system*.

Observe that for a null system δ and points p, q, if p is on $\delta(q)$ then $\delta^2(q) = q$ is on $\delta(p)$. The line joining p and q is fixed, since both $\delta(p)$ and $\delta(q)$ contain it. Conversely if the line joining r and s is fixed, it is $\delta(r) \cap \delta(s)$ and s is on $\delta(r)$. Now the lattice interval determined by two lines is the whole lattice if they are skew; if two fixed lines of δ meet in p, their lattice interval is $[p, \delta(p)]$. Hence the fixed lines form a submedium; medians (L, M, N) are obvious unless M meets N in p with L not on p, and then it is the intersection of $\delta(p)$ with the plane spanned by L and p. This is a 2-cell, for its edges are easily seen to satisfy the conditions of 2.13. We may as well call it the *null* (*system*) *2-cell* over k.

There is a simple, quite exceptional property that marks out the null 2-cells and many of the rudimentary 2-cells of 2.18: any three parallel intervals have an interval ticking all three. This property seems to deserve a name; call it *skewerability*.

2.19. *Null system 2-cells are skewerable, and a skewerable Paschian 2-cell having more than two points on some edge and more than two edges on some point is a null system 2-cell.*

PROOF. Edges of a null 2-cell in PG(3, k) = P are given by points of P. Three points p_1, p_2, p_3 give parallel edges when no two are on a fixed line, i.e. $p_i \notin \delta(p_j)$. Let π be a plane containing p_1, p_2, and p_3. Then $\delta(\pi)$ is on fixed lines with all p_i, so we have an edge meeting the given three, of course ticking them. (Three parallel singleton intervals also are ticked by an interval, namely the cell.)

The proof of the converse is a trifle clearer if we start with a 2-cell M whose dual is Paschian and skewerable (so that points of M will be points of P), and has three points on some edge and three edges on some point. We define the planes $\delta(p)$ of P as the sets of all points within an edge of a given point p. Accordingly lines of P are intersections of two planes $\delta(p) \cap \delta(q)$. If $\rho(p, q) = 1$, $\delta(p) \cap \delta(q)$ is $[pq]$. If $\rho(p, q) = 2$, the line is the set of points properly splitting p and q.

No two lines have two common points. This is clear if either line is an edge. For the other lines, *seams*, observe that two points on a seam are always at distance 2 and that when $\rho(p, q) = 2$ and r and s are two more points, r and s split p and q if and only if p and q split r and s. Suppose r and s split p and q and also p' and q';

we want every point t splitting p and q to split any two points of the seam $\delta(r) \cap \delta(s)$. It suffices to show that it splits p and p'. This follows from the Pasch axiom for M^*. Given the rectangle $prqs$, the point p' splitting one opposite pair of vertices, and t splitting the other, p' and t must be on a common edge, i.e. $\rho(p', t) = 1$. As we know $\rho(p, t)$ and $\rho(p, p')$, we have splitting.

Two points p, q lie on a line, an edge if $\rho(p, q) = 1$ and a seam if $\rho(p, q) = 2$. A plane contains a line if it contains two of its points. Three points lie on a plane, i.e. have a common neighbor; this is evident if two of them are on an edge, and otherwise it is skewerability of M^*. Similarly three planes have a common point. Having also sufficiently many points, we have a projective 3-space P. δ gives us a null system on P. If three points are collinear on L, their duals are coaxial, (i) on L if it is an edge, and (ii) on the seam of points splitting pairs of points of L, if a seam. So δ extends to lines (involutorily, fixing just edges). For a plane $\delta(p)$, put $\delta(\delta(p)) = p$; this is in fact the intersection of $\delta(q)$ over all points q of $\delta(p)$. The proof is complete.

2.20. *The dual of a null 2-cell over k is Paschian if and only if k has characteristic* 2; *then, it is skewerable if and only if k is perfect.*

Proof. If k does not have characteristic 2 then this cell M^* is "purely non-Paschian"; no rectangle has two intersecting crosses distinct from the sides. For that would give a 3 by 3 array of points; for each of its six diagonals, by skewerability of M, there is another point (of M^*) neighboring those three points; and the Pasch axiom for M shows that we have a subspace PG(3, 2) of PG(3, k).

In characteristic 2, the vertices of a rectangle of M^* may be assumed to have homogeneous coordinates as follows: $a = (1, 0, 0, 0)$, $b = (0, 0, 1, 0)$, $c = (0, 1, 0, 0)$, $d = (0, 0, 0, 1)$. The duality δ is now determined on those four points; for example, $\delta(a)$ contains a, b, and d, so it is $(0, 1, 0, 0)$. In characteristic 2, this determines the null system δ. For δ is given by a matrix S such that $SS^T = \lambda I$. The four known values give M the form

$$\begin{bmatrix} 0 & t_1 & 0 & 0 \\ t_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & t_3 \\ 0 & 0 & t_4 & 0 \end{bmatrix}.$$

But SS^T has t_i^2 along the diagonal, a constant; in characteristic 2, then, t_i is constant.

The general point on $[ab]$ is $(1, 0, r, 0)$; the cross from it meets $[dc]$ at $(0, r, 0, 1)$. Similarly the general cross the other way joins $(1, 0, 0, s)$ and $(0, s, 1, 0)$. They meet in $(1, rs, r, s)$.

Three parallels in M^* may be assumed to be $[ab]$, $[dc]$, as above, and $[xy]$ where $x \in \delta(a)$, $y \in \delta(b)$. (For $(a, x, y) \in \delta(a)$, etc.) Since $x \notin \delta(b)$ and $y \notin \delta(a)$, we may write $x = (r, 0, s, 1)$ and $y = (t, 1, u, 0)$. However, $y \in \delta(x)$, which makes $r = u$ (the characteristic being 2). A skewer of this kebab contains $z = (1, 0, v, 0)$

on $[ab]$ (or b itself), and $z' = (z, c, d) = (0, v, 0, 1)$ (or c). It contains $z'' = (z, x, y) = (r + vt, v, s + vr, 1)$, and $(z'', c, d) = (0, s + vr, 0, r + vt)$. That must be the same point as z', meaning $v(r + vt) = s + vr$, $v^2 = st^{-1}$. As s and t are arbitrary, this does not always happen unless the field is perfect. Even then we must also allow $t = 0$, but that puts y on $[bc]$, a skewer.

(xxx) A slight generalization of the null-system construction gives many pretty pictures of bipartite graphs, but no new 2-cells. Observe that the fixed elements of any selfduality δ of a modular lattice L form a submedium M. If L has length 4 then M has ρ-diameter at most 2 and is a 2-cell if it contains a rectangle. In this generality, perhaps all 2-cells embeddable in modular lattices can be represented this way. However, in a projective 3-space, the edges of M correspond to some of the points of the quadric surface (if δ is not a null system) S defined by $p \in \delta(p)$, namely those which are on at least two fixed lines. A fixed line $L = \delta(L)$ is contained in S. Three points on an edge means three concurrent coplanar lines in S, which therefore contains the plane–and we get no new 2-cell.

I have no idea whether the nonnull Paschian infinite 2-cells given by 2.20 are embeddable in modular lattices. Perhaps more accessible is the same question for the following example, which is also our first example with $e > 2$ points on each edge, $s > 2$ edges on each point, and $e \neq s$.

(xxxi) A nonskewerable Paschian 2-cell with 27 points and 45 edges. Call the edges A_0, A_1, A_2, A_3, A_4, and $1, 2, \ldots, 40$. The stars of points are $\{A_0, A_1, A_2, A_3, A_4\}$, the ten sets $\{A_i, 8i + j + 1, 8i + j + 2, 8i + j + 3, 8i + j + 4\}$ with $0 \leqslant i \leqslant 4$ and $j = 0$ or 4, and the following sixteen.

$$\begin{array}{llll}\{1, 9, 17, 25, 33\}, & \{1, 13, 21, 29, 37\}, & \{2, 10, 22, 30, 34\}, & \{2, 14, 18, 26, 38\},\\ \{3, 15, 19, 31, 35\}, & \{3, 11, 23, 27, 39\}, & \{4, 12, 20, 32, 40\}, & \{4, 16, 24, 28, 36\},\\ \{5, 9, 24, 31, 38\}, & \{5, 13, 20, 27, 34\}, & \{6, 16, 17, 30, 39\}, & \{6, 12, 21, 26, 35\},\\ \{7, 15, 22, 25, 40\}, & \{7, 11, 18, 29, 36\}, & \{8, 14, 23, 32, 33\}, & \{8, 10, 19, 28, 37\}.\end{array}$$

Checking conditions of 2.13 is straightforward. The cell is Paschian because there are only three points on each edge, which is easily seen to imply it. The parallels A_0, 9, 18 cannot be skewered.

This example contains a null 2-cell over Z_2 (edges A_0, A_1, A_4, 1, 2, 5, 8, 9, 10, 13, 14, 33, 34, 37, 38). If you consider such a 2-cell in a null 2-cell over the 4-element field, its 15 points lie on 30 more edges. (I.e. its 15 fixed lines have 30 new points.) The intersection pattern those edges would need to close up a cell is as given above. Over the 4-element field they do not intersect in that way. Indeed I would have thought that a 2-cell M not a bipartite graph could not have a proper subcell whose edges are whole edges of M, but we just refuted that. This example is not embeddable in any null 2-cell, since the dual is non-Paschian but contains a 3×3 array.

3. Extreme points. In a real vector space, an extreme point of a set S is a point not between two other points of S. In media, every point of a rectangle is between two others. We define an *extreme point* x of an edge-connected medium: any two edges containing x are both contained in some interval of the form $[ax]$.

3.1. *Distinct edges* $[xy]$, $[xz]$, *both contained in an interval* $[ax]$ *in a medium span a 2-cell.*

PROOF. The edges $[xy]$, $[xz]$ tick, so $x \in [yz]$. $(a, x, y) \neq x \neq (a, x, z)$, and $v = (a, y, z)$ has the same projections in $[xy]$ and $[xz]$ as a by 1.7. Call those y', z' respectively, and note $[y'z'] = [yz]$, $\rho(y', z') = 2$. Now $v \neq z'$ (since $(z', x, y) = x$); so $w = (y', v, z') \neq z'$. Also $w \neq y'$, for $y' \in [vz']$ would put x there, contrary to $z' = (v, x, z)$. Since w splits y' and z', $[wy']$ and $[wz']$ are edges by 1.16, $[y'z']$ is a 2-cell by 2.16, and it is $[yz]$, containing x.

We will not go into the nonedge-connected case beyond 3.1 and the remark (xxxii) in a taut medium, the points edge-connected to a given point a form an ideal. For this we need only that an interval $[bc]$ whose ends are edge-connected is an edge-connected medium, and 2.10 does that.

3.2. *Any finite number of edges in an edge-connected medium all containing a certain extreme point are contained in a cell.*

PROOF. Of course this is proved inductively, 3.1 taking care of two edges. Let us look explicitly at three edges $[xy_1]$, $[xy_2]$, $[xy_3]$. If one of the 2-cells $[y_iy_j]$ contains all the y's we are done. Otherwise let w_{ij} denote a fourth point of a frame, with x, y_i, y_j, for $i < j$ in $\{1, 2, 3\}$. Note, $[xy_3]$ ticks $[xw_{12}]$ and $\rho(y_3, w_{12}) = 3$. Since $\rho(y_3, w_{13}) = 1$ and $\rho(w_{13}, w_{12}) = 2$, we have $[w_{12}w_{13}]$ properly contained in $[w_{12}y_3]$. Hence $x \notin [w_{12}w_{13}]$. Since x is distant only 1 from y_1 in $[w_{12}w_{13}]$, $(x, w_{12}, w_{13}) = y_1$. Similarly each y_i is (x, w_{ij}, w_{ik}). So (w_{12}, w_{13}, w_{23}), being in all three of these intervals, is split from x by each w_i. It is distant only 1 from w_{13} and w_{23}; so they span a 2-cell. We showed $y_3 \notin [w_{12}w_{13}]$; similarly y_1 and y_2 are not in $[w_{13}w_{23}]$. Since they are distant only 1 from it, $[y_1y_2]$ is perspective with $[w_{13}w_{23}]$; by 2.14, we have a 3-cell.

For the inductive step, the construction will be no surprise, but we want a sharp formulation of the inductive hypothesis. Evidently we need only consider sets of edges $[xy_i]$ none of which is in the ideal spanned by the others. The proposition is that there are points y_S for all sets S of $s \leqslant n$ indices ($y_S = x$ for S empty and $= y_i$ for $S = \{i\}$) such that the 2^s points y_T, $T \subset S$, form a submedium isomorphic with the lattice of sets T indexing them ($T \mapsto y_T$ isomorphic) and span an s-cell. By 2.10, $T \mapsto y_T$ is then isometric (distance-preserving).

Having this for n, we construct y_Z for $Z = \{1, 2, \dots, n+1\}$ as (y_R, y_U, y_V) where $R = \{2, 3, \dots, n+1\}$, $U = \{1, n+1\}$, $V = \{1, 2, \dots, n\}$. Also put $W = \{2, \dots, n\}$ and observe that $[y_{n+1}y_W]$ is the n-cell $[xy_R]$. We will find it to be perspective with $[y_Uy_V]$, an n-cell an edge away; which will do it, since the same applies to any $(n+1)$-element set. Now $[xy_{n+1}]$ ticks $[xy_V]$, so $\rho(y_{n+1}, y_V) = n + 1$. Since $\rho(y_{n+1}, y_U) = 1$ and $\rho(y_U, y_V) \leqslant \rho(y_U, y_1) + \rho(y_1, y_V) = n$, we have $[y_Uy_V]$ properly contained in $[y_{n+1}y_V]$. Hence $x \notin [y_Uy_V]$. Then $y_W \notin [y_Uy_V]$ since $x \in [y_1y_W]$. Since $\rho(y_W, y_V) = 1$, $(y_W, y_U, y_V) = y_V$. But also $y_V \notin [y_{n+1}y_W]$ since x is there and $[xy_V]$ contains $y_1, \dots, y_n$. (Thus $y_1, \dots, y_{n+1}$ would be in that n-cell.) So $(y_V, y_{n+1}, y_W) = y_W$. Now these two intervals are disjoint, so y_{n+1} and y_U project to each other ($\rho(y_{n+1}, y_U)$ being 1), and perspectiveness is proved. Also x

and y_1 project to each other, so the $(n-1)$-cells $[xy_W]$, $[y_1y_V]$ are parallel. It follows that $y_Z \notin [y_1y_V]$; for that would imply $(y_Z, y_{n+1}, y_W) \in [xy_W]$, $[y_Ry_Z]$ meeting $[xy_W]$, hence containing $(y_R, x, y_W) = y_W$, and y_W also projecting to y_Z, distant at most $n-1$ from y_U. With $y_Z \notin [y_1y_V]$ and $\rho(y_Z, y_U) \geqslant \rho(y_V, y_U) - 1$ we know y_Z is distinct from all previously defined y_S; the y_S for $\{1\} \subseteq S \subseteq Z$ are isometric with the medium of sets S, and $[y_Uy_V]$ is an n-cell. Hence 2.14 applies and we have the required $(n+1)$-cell.

3.3. *An edge-connected medium whose distances are bounded is spanned by its extreme points.*

PROOF. Call x *remote* from a if x splits a and y only for $y = x$. If x splits a and $y \neq x$, $\rho(a, y) > \rho(a, x)$; so bounded distances implies that for any a and x, x splits a and some point remote from a. But if r is remote from a, r is extreme; indeed, for every edge $[rs]$, $(a, r, s) \neq r$ so $[rs] \subset [ar]$. Then consider one extreme point p and all the points r remote from p: a set of extreme points, and every point x is in some $[pr]$.

4. Reduced media. I can prove very little about reduced media. Unsurprisingly, 2.3–2.7 on parallels carry over. There is also a curious result on the Pasch property. The set P of points p of $[ab]$ whose crosses $[pq]$ in rectangles $abcd$ meet all crosses going the other way is closed under (1) taking splitting points and (2) intersecting edges, i.e. $q \in P$ if q lies in two different edges $[p_1p_2]$, $[p_3p_4]$, with the p_i in P. One may say, the non-Paschian part of a reduced interval is loosely stuck in. Illustrations: in the dual $[ab]$ of a null 2-cell over Z_3, (1) and (2) give a union (P) of two 4-point seams; the other 32 points lie in pairs in 16 edges having an end in each seam. In the dual of example (xxx), P consists of two 3-point seams; 27 more points lie in the 9 seam-to-seam edges, and there are 12 more points even more loosely linked to P.

4.1. *In any medium, if p splits a and b, and $[ab]$ is perspective with $[xy]$ as named, then p is in a cross of $[ab]$ and $[xy]$ which meets every ideal meeting $[ax]$ and $[by]$.*

PROOF. The proof of 2.3 shows that p is in a cross $[pq]$. An ideal of the sort in question contains a cross $[uv]$ of $[ax]$ and $[by]$. Now a to p to b to y is a path. We may assume $p \neq a$, and then $a \notin [py]$. Since $a = (p, a, x)$, $[py]$ contains no point of $[ax]$; $u \notin [py]$. (u, p, y) splits u and (u, p, q), so it is in a cross $[uw]$ of $[u(u, p, q)]$ and the perspective interval $[ap]$. As w is between p and (u, p, y), it is in $[py]$; being also in $[ap]$, $w = p$. Then (u, p, y) is (u, p, q). It is in $[uv]$ since $v = (u, b, y) = ((u, p, y), b, y)$, so $[pq]$ meets $[uv]$.

COROLLARY. *A reduced medium satisfies the axiom of Pasch if every 2-cell in it does.*

PROOF. We wish to show that the set P of points p of $[ab]$ such that for all perspective $[xy]$, any interval containing p and meeting $[xy]$ meets every cross of $[ax]$ and $[by]$, is all of $[ab]$. We know it contains $\{a, b\}$ and is closed under taking splitting points. $[ax]$ is another reduced interval. Suppose q is in an edge $[p_1p_2]$, $p_i \in P$, and consider the set Z_q of points z of $[ax]$ such that $[z(z, b, y)]$ meets $[q(q, x, y)]$. It contains $\{a, x\}$, it is closed under splitting by 4.1, it is closed under edges by the axiom of Pasch for 2-cells. So $Z_q = [ax]$. That is, P is closed under edges and therefore $P = [ab]$.

4.2. *Perspective intervals in a reduced medium are parallel and isomorphic.*

PROOF. If $[ab]$ and $[xy]$ are perspective as named, so are $[ax]$ and $[by]$. If $\rho(a, x) = 1$, the result is 2.13. Otherwise let p split a and x at distance 1 from a. Then p is in a cross $[pq]$ of $[ax]$ and $[by]$, perspective with $[ab]$ and $[xy]$. By 2.13, $[ab]$ and $[pq]$ are parallel, and since $\rho(p, x) < \rho(a, x)$ we may assume inductively that $[pq]$ and $[xy]$ are parallel. Any point u of $[ab]$ is in a cross $[uv]$ of $[ab]$ and $[pq]$, and $\rho(u, v) = 1$. Since a to p to x to y is a path, $a \notin [py]$, and $u \notin [py]$. Since u is within 1 of $v \in [pq] \subset [py]$, $v = (u, p, y)$. Then $[u(u, x, y)]$ ticks $[xy]$ and contains v, but no other point v' of $[pq]$ (since $(v', x, y) \neq (v, x, y)$) and therefore no other point but u of the parallel $[ab]$; each u is in a cross of $[ab]$ and $[xy]$. Since the assumptions on $[ab]$ and $[xy]$ were symmetric, they are parallel, and since ρ is preserved they are isomorphic.

4.3. *Propositions* 2.4 *and* 2.5 *are true in reduced media as well as in taut media.*

These are immediate from 4.2. Propositions 2.6 and 2.7 are virtually immediate, too. In an amalgam of reduced I and J along identified (Čebyšev) ideals, an interval $[xy]$, in the nontrivial case $x \in I, y \in J$, is built up by taking the splitting points $(x; J)$, $(y; I)$, and closing under paths of edges; so the amalgam is reduced. For the converse, observe that an edge-preserving bijection between two media preserves ρ, since $\rho(x, y)$ is just the number of steps within edges required to get from x to y. So if M is reduced and is covered by intersecting ideals I and J, then M and the amalgam have the same points, the same edges, and the same structure.

4.4. *Propositions* 2.6 *and* 2.7 *are true of reduced media as well as of taut media.*

The next observation applies to some nonreduced media. Let us call a medium *tame* if its perspective intervals are parallel. There is no need to add "and isomorphic"; parallelisms and their composites are edge-preserving bijections, so we can certainly conclude that they are isometric. This involves extending the definition of distance ρ, putting $\rho(x, y) = \infty$ if they are not edge-connected. Observe that 2.4 and 2.5 are true in tame media–same proofs. Also, parallels are equidistant.

4.5. *In a tame medium, if intervals I and J tick at p and I' is parallel to I at distance d, then the projection of I' in J lies within distance d of p.*

PROOF. If $d = \infty$ this is vacuous. If d is finite so is the distance e between $(I'; J)$ and $(J; I')$, since I' contains a point distant d from $p \in J$. For any $r \in (J; I')$, let $q = (r; I)$ and $s = (r; J)$. Then $\rho(q, s) \leqslant \rho(q, r) + \rho(r, s) = d + e$. But p splits q

and s, so $d + e \geqslant \rho(q, p) + \rho(p, s)$. On the other hand both q and s split r and p; $\rho(q, p) + d = \rho(p, s) + e$. Adding, $2d \geqslant 2\rho(p, s)$; every s in $(I'; J)$ is within d of p.

For the next result let us say, given a tame rectangle $abyx$, that a point of one of its sides, say $[ab]$, is *Paschian in abyx* if the cross of $[ab]$ and $[xy]$ containing it meets every cross of $[ax]$ and $[by]$.

4.6. *In a reduced rectangle abyx the set of Paschian points in* $[ab]$ *contains all points that split pairs of Paschian points and all intersections of two edges that join pairs of Paschian points.*

Proof. The first assertion is immediate from 3.1. For the second, it suffices if given an edge E joining Paschian p_1, p_2, in $[ab]$, and a non-Paschian point $p_3 \in E$, we succeed in constructing E from p_3 and the rectangle. To this end consider the set R of points r of $[ax]$ whose crosses meet the cross $[p_3q_3]$ of $[ab]$ and $[xy]$. Glance at the proof of 4.1; it shows that such sets as R are closed under splitting. Since p_3 is not Paschian in $abyx$, R omits some point r_3 of an edge $[r_1r_2]$, r_1 and r_2 in R. Project $[p_3q_3]$ upon the cross $[r_3s_3]$ of $[ax]$ and $[by]$. By 4.5, the image I lies within distance 1 of the point where $[r_3s_3]$ meets the cross through p_1, and also within distance 1 of the intersection with the cross through p_2; thus it is a subset of an edge. On the other hand, intersecting with the 2-cell given by the p_1, p_2, r_1 and r_2 crosses, the p_3 and r_3 crosses reduce to edges which are disjoint, hence parallel; I is that edge. The parallelism from $[r_3s_3]$ to $[ab]$ takes it to E.

One might gladly trade information on the Paschianness of reduced media for information on their tautness. Indeed, if they are all taut, then we know almost nothing about nontaut media, and might live for years without knowing more. There are plenty of examples, of course. We turn now to a construction of no value that I know of except fertility. In that line, it gives 36 media on a five-point set corresponding to the same reduced medium.

By the way, (xxxiii) no such thing happens with cells; a medium isometric with an n-cell is an n-cell. This is obvious for $n = 2$. The inductive proof turns on the $(n - 1)$-faces $[ay]$ of the cell being ideals in the other structure. For any $x \notin [ay]$ there is a frame having (x, a, y) as a vertex, and the rest is very easy.

We define a *sprawl* as a set S with a family of subsets $\mathcal{G}$ closed under intersection (this includes empty intersection: $S \in \mathcal{G}$) and including each set that contains elements of $\mathcal{G}$ containing each of its two-point subsets. (Thus empty and singleton sets belong to $\mathcal{G}$.) This would seem to be the most general idea one could have of a set with a notion of betweenness. In a sprawl, of course, the points *between* x and y are the elements of the smallest element $|xy|$ of $\mathcal{G}$ that contains $\{x, y\}$.

An *embedding* of sprawls is an injection i: $S \to S'$ satisfying $x \in |yz|$ if and only if $i(x) \in |i(y)i(z)|$.

(xxxiv) Every sprawl can be embedded in a product of edges. Construction: for each $I \in \mathcal{G}$ form the quotient set S/I collapsing I to a point, make S/I an edge E_I, and map S to the product of all E_I using the quotient maps $S \to E_I$. The simple verification is left to the reader.

That construction is fairly analogous to embedding an arbitrary partially ordered set in a Boolean algebra. If one asks for preservation of such medians as happen to exist in a sprawl, one is into an area doubtless more interesting, but not our concern here. Here we will note a more economical embedding of sprawls in media. Define the *cone* CS over a sprawl S as a set consisting of S and one more point 0, with a family of subsets $C\mathcal{I}$ consisting of the null set, the singletons, and the sets $I \cup \{0\}$ where $I \in \mathcal{I}$.

(xxxv) The cone over any sprawl is a medium. What must be checked is (from 1.3) that it is a sprawl, which is quick and easy, and that it has medians (x, y, z). The median is $0 \in |yz|$ in all nontrivial cases.

(xxxvi) Cones over nonisomorphic sprawls are nonisomorphic media. (Indeed, the vertex 0 is determined as the intersection of all nonsingletons in $C\mathcal{I}$ except for a two-point cone.) The number of sprawls on a four-point set is 36 (8 media, 28 not). In 33 cases the cone is nontaut, in consequence of:

4.7. *A taut cone is embeddable in the lattice of subspaces of a projective plane as* 0 *and a set of points.*

Proof. The base S of the cone cannot have two comparable nondegenerate intervals. For if $|uv|$ is properly contained in $|xy|$, there is another such pair of intervals having a common endpoint–by replacing v with x unless the result, $|ux|$, contains y, in which case we have $|ux|$ properly containing $|uv|$. Thus we have b in $|ac|$, $c \notin |ab|$, $a \neq b$. Since $(c, a, b) = 0$, 2.1 fails.

But now two intervals cannot have two common points. So the points and nonsingleton intervals of S form a partial projective plane. Hall showed that such a thing is embeddable in a projective plane [7]. Inspection shows that, taking 0 to 0, we have an embedding of media.

By (xxvi), the set of points in a projective plane is arbitrary (and a set of points in a higher projective space will do as well; the medium structure simply loses the geometry beyond collinearity).

5. Isotropy. Call a medium M *isotropic* if projections upon intervals in M are homomorphic:

$$((x, u, v), (y, u, v), (z, u, v)) = ((x, y, z), u, v). \tag{I}$$

5.1. Theorem. *A medium is isotropic if and only if* $w \in [xy]$ *implies* $w \in [(x, w, z)(y, w, z)]$.

Proof. If projection upon $[wz]$ is homomorphic it takes (w, x, y) to $(w, (x, w, z), (y, w, z))$; if (w, x, y) is w, it is fixed. Conversely, assume this condition. It implies tautness (put $z = x$); so perspective intervals are parallel.

We show first that projecting $[xy]$ to $[uv]$ takes $t \in [xy]$ to a point between (x, u, v) and (y, u, v). By 2.5, the image J of $[xy]$ is a Čebyšev ideal, and it is parallel to its projection K back in $[xy]$. Note also that $((t; J); K)$ is $(t; K)$, for $[t(t; J)]$ contains only one point of J, and thus only one point of the parallel ideal K. This holds for every point of $[xy]$, even the ends.

We need only show that $(t;\, K)$ is between $(x;\, K)$ and $(y;\, K)$. If it were not, it would not be between $(x;\, K)$ and $((y;\, K), (t;\, K), (x;\, K))$ (a subinterval). But by 1.8 this means $(t;\, K) \notin [(x, (t;\, K), (x;\, K))(y, (t;\, K), (x;\, K))]$, contrary to hypothesis.

Knowing that betweenness is preserved, consider x, y, z, u, v of (I), $y^* = (y, u, v)$, $y' = (y^*, y, z)$, and z^*, z' defined similarly. $[y^*z^*]$ and $[y'z']$ are parallel, by 2.5 again. Since (x, y, z) is between x and each $t \in [yz]$, $((x, y, z), u, v)$ is between (x, u, v) and (t, u, v). Since each element of $[y^*z^*]$ has this form ($t \in [y'z']$), the proof is complete.

(xxxvii) Isotropy is characterized by (a) projections upon intervals preserve betweenness, or (b) (I) specialized by $u = x$. For each of these conditions is implied by (I) and implies the condition of 5.1. Also (5.2, 5.8) projections upon Čebyšev sets are homomorphic.

5.2. *In an isotropic medium all Čebyšev sets are ideals.*

Proof. Let S be a Čebyšev set containing x and z, and let $y \in [xz]$. Let $u = (y;\, S)$. Projecting to $[yu]$, S goes into itself by 1.6. But S ticks $[yu]$ at u, so x and z can only go to u. Preservation of betweenness means $y \in [uu]$, $y \in S$.

Let us call a medium in which $(x, y, z) = x$ whenever $y \neq z$ *indiscrete*. Of course these are just the edges and the media with fewer than two points.

5.3. *Every subdirectly irreducible isotropic medium is indiscrete.*

Proof. If M is not indiscrete, we need to show that any distinct x, y are separated by projection into a proper subinterval of M. $[xy]$ will do it if it is an edge. If not, then it has a nondegenerate proper subinterval I, and the betweenness-preserving projection upon I cannot collapse $\{x, y\}$.

5.4. Theorem. *Every isotropic medium is a subdirect product of indiscrete media.*

This follows from 4.3 by a standard theorem of Birkhoff [**2**].

(xxxviii) Of course it follows that isotropic media are embeddable in modular lattices, are Paschian, satisfy all identities valid in indiscrete media. Note that they satisfy all identical implications valid in indiscrete media, since those too carry over to subdirect products.

5.5. *For intervals in isotropic media, parallelism is a transitive relation and composites of parallelisms are parallelisms.*

Proof. This is true in indiscrete media and (since parallel = perspective) expressed by identical implications.

5.6. *For Čebyšev sets in isotropic media, the composite* (*in either order*) *of a projection and a parallelism is a projection.*

Proof. The Čebyšev sets are ideals (5.2). The proposition that y is the projection of x in a Čebyšev ideal I containing y is equivalent to y's being the projection of x in $[yz]$ for each $z \in I$. Thus the claim reduces to the case of intervals. As in 5.5, it need only be checked for edges, and it checks.

(xxxix) Any composite of projections has the same values (but not the same codomain) as a projection; specifically $((x;\ I);\ J) = ((x;\ I);\ (I;\ J)) = (((x;\ I);$ $(J;\ I));\ (I;\ J))$ by 2.5 and 5.6, which is $((x;\ (J;\ I));\ (I;\ J))$ by 1.8, $(x;\ (I;\ J))$ by 5.6.

5.7. *In an isotropic medium, the projection of an interval in a Čebyšev ideal is an interval.*

PROOF. By 2.5 the projection of $[xy]$ is an ideal J, and J is parallel to its projection K in $[xy]$. It remains to show that K is an interval, i.e. each $u \in K$ is between $x^* = (x;\ K)$ and $y^* = (y;\ K)$. We have $u \in [xy]$ and $y^* = (y, y^*, u)$. To show that these equations imply $u \in [xy^*]$, it suffices to check in an edge. There $u \notin [xy^*]$ means $x = y^*$; so $x = (y, x, u)$. But now $[xy]$, containing u, ticks $[xu]$, so $x = u$, a contradiction. Finally, replace u, x, y, y^* in the last bit by u, y^*, x, x^* respectively. Still $u \in [y^*x]$ and $x^* = (x, x^*, u)$, so again $u \in [y^*x^*]$.

5.8. *In an isotropic medium, projection upon a Čebyšev ideal is homomorphic.*

PROOF. The proposition $p = (x, y, z)$ is equivalent to the conjunction: $[xp]$ ticks $[pq]$ for all $q \in [yz]$ (given $p \in [yz]$). Similarly after projection, $[yz]$ going to an interval. The projection takes $[xq]$ to an interval $[x^*q^*]$, and on $[xq]$ it agrees with projection to $[x^*q^*]$, a homomorphism.

In a product of edges, an ideal is a partial product (i.e. some coordinates are completely determined, the rest unrestricted); so there is a unique parallel to a given ideal through a given point. Something like that holds in all isotropic media. For non-Čebyšev ideals, what is involved is not parallelism, and we leave it for the interested reader.

5.9. *For a Čebyšev ideal I in an isotropic medium,* $[xy]$ *is parallel to* $[(x;\ I)(y;\ I)]$ *if and only if every homomorphism constant on I is constant on* $[xy]$.

PROOF. For two parallel intervals in any medium, $[ab]$ and $[xy]$, a homomorphism nonconstant on $[xy]$ is nonconstant on $[ab]$, since $u \in [xy]$ is recoverable as $((u, a, b), x, y)$. Hence "only if". Conversely, if $[xy]$ is nonparallel to its projection in I then one of its points, z, is not $((z;\ I), x, y)$. Then projection to $[z(z;\ I)]$ is a homomorphism with constant value $(z;\ I)$ on I but nonconstant on $[xy]$.

Let us call an ideal J a *partial parallel* (or partially parallel) to an ideal I if the crosses of I and J cover J.

5.10. *For a point x and a Čebyšev ideal I in an isotropic medium there is a largest partial parallel to I containing x. These partial parallels partition the medium and are the equivalence classes of a congruence relation.*

PROOF. Let J_x be the set of all y such that $[xy]$ is partially parallel to I. By 5.9, this is an ideal. Since every interval in J_x is (by 5.9) partially parallel to I, so is all of J_x, and clearly it contains any partial parallel to I through x. Hence if J_u and J_v have a common point z, both are contained in J_z and hence equal to J_z: a partition. From 5.9 again, if $x' \in J_x$, $y' \in J_y$, $z' \in J_z$, then $[(x, y, z)(x', y', z')]$ is partially parallel to I, and the proof is complete.

5.11. *In isotropic media,* $r \in [ps]$ *and* $t \in [ru]$ *imply* $t \in [s(t, p, u)]$.

PROOF. We need only check in an edge. Falsity of the conclusion, there, means $s = (t, p, u) \neq t$. That implies $p = u \neq t$, $s = p$; from $r \in [ps]$, $r = p$, and now $t \in [ru]$ is violated.

5.12. THEOREM. *The ideal join of nonempty ideals* I, J, *in an isotropic medium is the set of all* (x, i, j), $i \in I$ *and* $j \in J$.

PROOF. Obviously this is true if that set is an ideal. That is, given p and p' in I, s and s' in J, $r \in [ps]$, $r' \in [p's']$, $z \in [rr']$, we want the following. 5.11 on $r \in [ps]$, $z \in [rr']$ gives z between s and (z, p, r'). Again, on $r' \in [p's']$, $(z, p, r') \in [r'p]$ it gives (z, p, r') between s' and $((z, p, r'), p', p) = p'' \in I$. Don't tire! On $(z, p, r') \in [s'p'']$ and $z \in [(z, p, r')s]$, it gives z between p'' and $(z, s', s) \in J$, as required.

COROLLARY. *The ideal join of two Čebyšev ideals is Čebyšev.*

PROOF. In fact the projection of p in the join of I and J is (p, i, j) where $i = (p; I)$ and $j = (p; J)$. By the theorem, we need only the conclusion $(p, (p, i, j), z) = (p, i, j)$ for each $z \in [xy]$, $x \in I$ and $y \in J$. We have $(p, i, x) = i$, $(p, j, y) = j$. So from three identities we want an identity. If it does not follow, it fails in some edge, giving $(p, i, j) \neq z$ and $\neq p$. Therefore $i = j \neq p$. But $(p, i, x) = i \neq p$ makes $i = x$; also $(p, i, y) = (p, j, y) = i$, y equals i (and j, and x). Then $z \in [xy]$ is another name of this same point, as is (p, i, j), a contradiction.

5.13. *Two disjoint ideals in an isotropic medium* M *can be separated by a homomorphism* h: $M \to N$ *constant on each of them.*

PROOF. Two disjoint Čebyšev ideals have a cross $[ab]$ and a homomorphism $(\ ;[ab])$ as indicated. From the corollary to 5.12, finitely spanned ideals are Čebyšev. So two disjoint ideals I, J can be represented as directed unions of Čebyšev ideals I_α, J_β separated by suitable $h_{\alpha\beta}$: $M \to N_{\alpha\beta}$. Pass to an ultraproduct.

COROLLARY. *Every ideal is a fiber* $h^{-1}(p)$ *of a homomorphism* h: $M \to N$.

(xl) N in 5.13 can of course be taken indiscrete. The fibers of homomorphisms to indiscrete media are characterized as those ideals I such that for some ideal J, I is maximal disjoint from J. These are a basis (intersection-basis) for the ideals.

Note, 5.13 fails completely in every nondistributive modular lattice; for there is a five-element sublattice all of whose edges are transparallel, and any medium homomorphism on it is injective or constant.

5.14. *Finitely generated isotropic media are finite.*

For this it suffices to examine the free ones on n generators, which evidently are constructed by taking all possible functions from an n-element set onto an indiscrete medium (of n or fewer elements). It is routine to check (xli) that the free taut medium on three generators, described after 2.2, is isotropic.

5.15. *In isotropic media, finite edges are projective.*

PROOF. If h: $M \to E$ is surjective, E an n-point edge and M isotropic, consider sets S of n points $p_1, \ldots, p_n$ of M mapping onto E. We want one for which all intervals $[p_ip_j]$, $i \neq j$, contain S. If that fails for i, j, project S upon $[p_ip_j]$. Since $(h(x), h(p_i), h(p_j)) = h(x)$ identically, the image S' still maps onto E. Since the projection is homomorphic all betweenness relations true in S are true in S', and also $x \in [p_ip_j]$ for all x in S'. Finitely many applications of this give S^k mapping isomorphically upon E.

An infinite edge is not projective, being a quotient of a bouquet of finite edges corresponding to its finite subsets.

5.16. *The proper subvarieties of the isotropic media are one for each positive integer n, consisting of the submedia of powers of an n-element indiscrete medium.*

PROOF. If a subvariety contains arbitrarily large finite edges it contains all finite edges, bouquets of them, all edges, thus all isotropic media. If it contains a largest edge (or singleton) E it contains submedia of powers of E; but every member M of the variety is a subdirect product of edges, quotients of M, which hence belong to the variety and are submedia of E. Finally, no finite edge is in the subvariety generated by smaller edges since it is projective and not a subdirect product of them.

REMARK. Some specific laws for the smallest of these varieties (except the trivial one) will be examined shortly. For the next one up, all I can make of the laws is ugliness.

5.17. *An edge-connected medium is isotropic if and only if it is reduced and parallelism of intervals is transitive.*

PROOF. The conditions are necessary (5.2, 5.5). If M satisfies them, we wish to show that projection to $[uv]$ preserves betweenness. By 1.21 we need only (i) the image of an edge $[ab]$ lies in $[(a, u, v)(b, u, v)]$, and (ii) s splitting a and b goes to (s, u, v) in $[(a, u, v)(b, u, v)]$. But (i) always holds by 1.12.

For (ii) it suffices to treat the case $\rho(a, s) = 1$ (by induction). Let $a^* = (a, u, v)$, $a' = (a^*, a, b)$, b^* and b' similarly. $[a^*b^*]$ and $[a'b']$ are perspective as named, so by (xviii) $\rho(a^*, b^*) = \rho(a', b')$. $[aa']$ ticks $[a'b']$; for a subinterval I of $[a'b']$ properly containing $\{a'\}$ contains $(b'; I) \neq a'$, splitting a' and b', closer to b', projecting to a^* if $I \subset [aa']$, a violation of (xviii). Similarly $[bb']$ ticks $[a'b']$, so there is a path of edges Π from a through a' and b' to b. By 1.19, $[as]$ is parallel with an edge E in Π.

Assume $s^* = (s, u, v) \notin [a^*b^*]$. Then $s^* \neq a^*$, and $[a^*s^*]$ is an edge parallel to $[as]$ and thus to E. But the projection of E in $[uv] \supset [a^*s^*]$ is in $[a^*b^*]$, a contradiction.

Finally, we have here, as in affine spaces:

5.18. *In the lattice of ideals of an isotropic medium, each interval not containing* 0 *is modular.*

PROOF. We must show that there is no nonmodular 5-element sublattice. Thus if $I \supset J$ properly and they have the same join with K and $M = J \cap K$ is nonempty, we want a point p of $I \cap K$ not in M. There is $i \in I$ not in J. By 5.12, $i \in [jk]$ for some $j \in J$, $k \in K$. Take $m \in M$ and $p = (k, i, m)$; that is in $[km] \subset K$ and in $[im] \subset I$. But by 5.1, since i is between j and k it is also between (j, i, m) and p. Of course $(j, i, m) \in [jp]$, so $i \in [jp]$. Since $i \notin J, p \notin J$.

COROLLARY. *The lattice of ideals of an isotropic medium, if of finite length, satisfies the Jordan-Dedekind chain condition.*

PROOF. By 5.18 we need only consider two maximal chains from 0 to I. They contain singletons p, q respectively, and by 5.18 the chain from p to I has the same length as a similar chain going through $[pq]$. Any one; and so for the other chain. But from p to $[pq]$, in an edge-connected medium, every ideal is Čebyšev; by isotropy, it is an interval (5.7), and the number of steps is $\rho(p, q)$. The same for q.

6. Symmetry. By 5.16, the smallest nontrivial subvariety of the isotropic media consists of the subdirect powers of the two-element indiscrete medium **2**. That is the medium underlying the two-element lattice, so (by Stone's representation theorem) the variety consists of the media embeddable in distributive lattices. By the way, it is the smallest nontrivial variety of media, since any two elements of any medium form a submedium **2**. We call these media *symmetric*.

The algebra of the median operation (x, y, z) in distributive lattices was studied by Birkhoff and Kiss [3], who noted the laws (1), (2), symmetry $(x, y, z) = (y, z, x)$ (with (1), making (x, y, z) a symmetric function of its arguments), and

$$((x, y, z), u, v) = ((x, u, v), y, (z, u, v)). \qquad (*)$$

The main theorem of [3] is, in the present terminology, that intervals in such an algebra are distributive lattices. Of course these algebras are precisely the symmetric media. Birkhoff and Kiss showed that they include the subsets of distributive lattices closed under median, i.e. the symmetric media. For the converse, these laws give (3), (4), (5) by straightforward simplification of the left sides, and in (3) of the right side too (using symmetry). So we are in media. At this point perhaps the simplest way to finish is to prove isotropy by

$$((x, u, v), (y, u, v), (z, u, v)) = ((x, u, v), (y, u, v), ((z, u, v), u, v))$$

((5) and symmetry), which is $((x, (y, u, v), (z, u, v)), u, v)$ by $(*)$, then $(((x, y, z), u, v), u, v)$ by $(*)$ and symmetry, and $((x, y, z), u, v)$ by (5) and symmetry; and to use 4.3 and note that edges larger than **2** do not satisfy symmetry.

6.1. *The algebras of Birkhoff-Kiss* [3] *are the symmetric media.*

Of course there are several simple or interesting characterizations of symmetric media. Among media, surely the simplest (and one that should soon be proved, to justify our terminology) is symmetry. The proof of 6.1 used $(*)$ too, so we must go around it.

6.2. *In a medium satisfying symmetry, every $p \in [xy]$ splits x and y.*

PROOF. $p = (p, x, y) = (x, y, p)$ splitting x from everything in $[yp]$.

6.3. *A medium satisfying symmetry is a symmetric medium.*

PROOF. By the proof of 6.1, we need only show it is isotropic. Use the criterion of 5.1. $w \in [xy]$ splits x and y; so the concatenation of paths $[x(x, w, z)]$, $[(x, w, z)w]$, and $[w(y, w, z)]$, $[(y, w, z)y]$, is a path. Thus w splits (x, w, z) and (y, w, z).

Remark, the proof of 6.3 shows that the conclusion of 6.2 implies isotropy. It is also false in edges of more than two points, so it is another characterization of symmetric media. It can be modified to give a reasonable symmetric version of Theorem 1.3, viz. 6.5. Note, as corollary of 2.9:

6.4. *In a symmetric medium, three intervals* $[xy]$, $[xz]$, $[yz]$ *have exactly one common point.*

6.5. *The ideals of a symmetric medium are an arbitrary family of subsets* $\mathcal{I}$ *closed under intersection, including every set which contains elements of* $\mathcal{I}$ *containing each pair of its points, and such that for all* x, y, z *there is a unique point* (x, y, z) *belonging to every member of* $\mathcal{I}$ *which contains two of* x, y, *and* z.

PROOF. In a symmetric medium we have these properties. Conversely, a family of subsets with these properties gives us a ternary operation symmetric in its arguments. The intervals $[yz]$, smallest element of $\mathcal{I}$ containing $\{y, z\}$, satisfy "$x \in [yz]$ implies $(x, y, z) = x$", since x certainly belongs to $[xy]$ and $[xz]$ also. So (2) and (3) hold too. The left side of (4) is $((y, u, v), (x, u, v), x) = (P, Q, R)$ where two of P, Q, R lie in $[xu]$, two in $[xv]$, two in $[uv]$; hence it must be (x, u, v), the right side. Similarly the left side of (5) is the median of three points of $[xy]$ with two of them in $[xz]$ and two in $[yz]$. Thus we have a medium satisfying symmetry; since $\mathcal{I}$ gives the right intervals, it is the set of ideals.

Fifth characterization:

6.6. *Symmetric media are defined by identities in four variables, specifically* (1), (2), (4), *and symmetry.*

PROOF. $((t, u, v), u, v) = ((u, t, u), v, (v, t, u))$ by total symmetry and (2), which is (v, t, u) by (4), which is (t, u, v). So we have (i); the interval $[uv]$ is the set of all (x, u, v). With symmetry, (5) is immediate. Now given c in $[bd]$, b and d in $[ae]$, (4) yields $((b, a, e), c, (c, a, e)) = (c, a, e)$; this, $b \in [ae]$, and symmetry give $c^* = (c, a, e) \in [bc]$. From symmetry and (4), $(c^*, d, c) = ((c^*, b, c), d, (d, b, c)) = (d, b, c) = c \in [c^*d]$. But exchanging b and d, c^* is also in $[cd]$. Then $c = (c, c^*, d) = c^* \in [ae]$. Thus the right side of (3) is in $[uv]$ and equal to the left side. By 6.3, the proof is complete.

(xlii) Media are not defined by four-variable laws. There is a six-element ternary algebra on $\{a, b, c, d, e, 0\}$ whose 216 values of (x, y, z) are defined, 36 by (2) (viz. $(x, x, y) = x$), 30 more by $(x, y, x) = x$ (with $y \neq x$), 3 by $(b, a, e) = b$, $(d, a, e) = d$, $(c, b, d) = c$, 3 more by applying (1) to those, and the remaining 144 values are 0. What this is, in effect, is the "cone" over $\{a, b, c, d, e\}$, which is not quite a sprawl. Anyway the 5 subsets complementary to the singletons $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$ are subalgebras each of which is the cone over a sprawl, hence a medium. Every 4-element subset is contained in one of these.

(xliii) From 6.4, a betweenness-preserving mapping from a medium to a symmetric medium is a homomorphism; and from the converse of 6.4, this is another characterization. Seventh characterization (exercise): in each interval $[ab]$ the relation $p \in [aq]$ is equivalent to $q \in [pb]$.

Why is everything a characterization? Because the nice, small varieties of media are totally ordered; so noncharacteristic good features of symmetric media generalize to the variety V_3 of submedia of powers of a 3-point edge.

(xliv) V_3 is not the only next-to-minimal variety of media. It may, as far as I know, be the only such variety of taut media. But in the smallest variety W containing a cone over the totally ordered set **3**, it is easy to see from the law

$$((x, y, z), (y, z, x), (z, x, y)) = ((y, z, x), (z, x, y), (x, y, z))$$

that every nonsymmetric element contains a cone over **3**; so the only proper subvariety is the smallest variety, the symmetric media.

(xlv) The elementary theory of symmetric media is undecidable. For distributive lattices, the like result is proved in **[5]**, and according to **[4]**, proved earlier by Grzegorczyk **[6]**. Thence easily distributive lattices with 0 and 1, intervals in symmetric media, symmetric media.

(xlvi) Edge-connected symmetric media are describable as graphs, the vertices and edges of the graph being the points and edges of the medium. In terms of the obvious distance ρ, intervals $[xy]$ being unions of shortest paths, it is easy to check that the graphs are precisely those connected ones in which three intervals $[xy]$, $[xz]$, $[yz]$ always have a unique common vertex.

The description in (xlvi) sounds simple, but the crucial property of the graph is not a convenient one. One can give a much clearer description by means of amalgams. Call a medium *indecomposable* if it is not an amalgam of two of its proper ideals. By 2.7, for taut media, this means that it is not covered by two intersecting Čebyšev proper ideals. (The cone over **3** shows that this is not equivalent for arbitrary media.) Note, every symmetric medium of more than one point is covered by two proper ideals, fibers of a homomorphism onto **2**. Obviously every finite medium can be built up by amalgamation from indecomposable ideals.

6.7. *An edge-connected symmetric medium is indecomposable if and only if it is empty or isomorphic with the* (*lattice*) *medium of all finite subsets of some set.*

Proof. First the equivalence, for these media, of (1) indecomposability, (2) every interval $[xy]$, regarded as a lattice as in 1.16, is complemented, and (3) every interval of length 2 is complemented. If M decomposes into (Čebyšev) ideals M_1, M_2, with $x \notin M_2$ and $y \notin M_1$, then $(x; M_2) = f$ has no complement c in $[xy]$. For $[fc]$ would have to be $[xy]$; but if $c \in M_1, y \notin [fc] \subset M_1$, and symmetrically. Next, if a finite-length distributive lattice is uncomplemented not all its subintervals of length 2 are complemented. Finally, an uncomplemented interval of length 2 is an ideal **3**; project onto it, and there is a decomposition (since every ideal is Čebyšev).

The indicated media clearly satisfy (2). On the other hand, given (2), pick any vertex and call it 0. Define $x \leqslant y$ if $x \in [0y]$. Any two elements x, y, have a supremum, the complement of 0 in $[xy]$. In view of (2), clearly we have the medium of finite subsets of the set of neighbors n of 0 (defined by $\rho(0, n) = 1$).

From 6.7, clearly, a finite symmetric medium M can be built up by amalgamation from media of all finite subsets of various sets (if we except the empty medium; some might prefer to call it a finite amalgam of 0 constituents) which are ideals of M; call those *blocks*. As for infinite M, 6.7 tells us at most how to take them apart. They can also be put together.

6.8. LEMMA. *If a proper ideal I in an edge-connected symmetric medium M contains a block which is maximal in M, then there exist a proper ideal I_0 of I and an ideal J isomorphic with $I_0 \times \mathbf{2}$ such that $I \cup J$ is an ideal and $I \cap J = I_0 \neq J$.*

PROOF. I is nonempty and $\neq M$, so there is $x \notin I$ at distance 1 from $(x; I)$. Let I' be the largest partial parallel to I through x. The projection I_0 of I' in I is an ideal parallel to I' at distance 1, and $J = I' \cup I_0$ is an ideal as follows. Consider r between $p \in I'$ and $q \in I_0$. If $r \notin I$, as it splits p and q and $(r; I)$ splits r and q, thus r splits p and $(r; I)$. So does $(p; I)$. Since projection decreases distances, $\rho(r, (r; I)) = \rho(p, (r; I)) - \rho(p, r) \leqslant \rho(p, (p; I)) = 1$, whence equality and parallelism; $r \in I'$. But if $r \in I$ it splits $(p; I)$ and q, so it is in I_0.

Evidently J is isomorphic with $I_0 \times \mathbf{2}$, and $I \cap J = I_0 \neq J$. Further, for r between $p \in J$ and $q \in I$, if $p \in I'$ the proof above applies to show $r \in I'$ or $r \in I$; otherwise $p \in I_0 \subset I$ and $r \in I$. And finally, $I_0 \subset I_0 \times \mathbf{2}$ contains no maximal block of M, so if I does then $I_0 \neq I$.

6.9. THEOREM. *Each edge-connected symmetric medium M is the union of a directed set of its ideals I_d in which* (0) *for $d \neq e$, $I_d \neq I_e$,* (1) *every totally ordered subset is well-ordered,* (2) *the minimal elements are blocks of M, and* (3) *a non-minimal element either has just two immediate predecessors and is their amalgam, or has directed proper predecessors and is their union.*

PROOF. We construct an expanding sequence of ideals I^α beginning with some maximal block I^0 of M. As long as I^α is a proper ideal, we apply the lemma to give $J^\alpha = I_0^\alpha \times \mathbf{2}$ and the amalgam $I^\alpha \cup J^\alpha = I^{\alpha+1}$. At a limit ordinal λ, I^λ is the union of the previous I^α. Finally we arrive at $I^\sigma = M$.

We may use induction on $\sigma = \sigma(M)$, the least ordinal such that some such sequence leads to $I^\sigma = M$. For nonempty ideals K of M, $\sigma(K) \leqslant \sigma(M)$. For let I^α be the first of these I's that meets K. If $\alpha = 0$, $I^\alpha \cap K$ is a block, $\sigma(I^\alpha \cap K) = \alpha$. If $\alpha > 0$ it is $\beta + 1$ for some β (since I^λ is just a union) and $I^\alpha \cap K$ is partially parallel to I^β; hence $\sigma(I^\alpha \cap K) \leqslant \beta < \alpha$. Now an evident induction establishes $\sigma(K) \leqslant \sigma(M)$.

The proof is nearly finished by observing that the truth of 6.9 for I^α and for I_0^α implies it for $I^{\alpha+1}$ (nonduplication (0) holding because in J^α we use only blocks $B \times \mathbf{2}$ not contained in I^α); it remains to say correctly "act smoothly at limit ordinals". That is, we run induction on the proposition that for $M = I^\sigma$ and $\tau < \sigma$, any directed set D_τ validating 6.9 for I^τ can be extended to one for M. If this were false, by previous remarks a first counterexample must be at a limit ordinal σ.

Construct directed sets D_ρ for $\tau \leqslant \rho < \sigma$ extending D_τ, getting $D_{\alpha+1}$ as an extension of D_α and D_λ (λ a limit ordinal) by union. Then the union of these is the required D_σ.

The theorem certainly says that M can be constructed by amalgamating together some sets of its (distinct) blocks; but it leaves open a couple of questions. First, are all amalgams of symmetric media symmetric? (xlvii) Yes; by 6.3 and the explicit description of (x, y, z) two sentences before 2.6.

Second, how many blocks are needed, say in a finite symmetric medium M? Since each block in an amalgam of I and J is (as is easily seen) already contained in I or in J, each maximal block of M must occur as a constituent. One might think that should suffice, but (xlviii) No. The example has 111 elements, and 15 maximal blocks, all of which are 4-cells. First we describe $K \subset \mathbf{3}^4$ by giving its 9 maximal blocks. In rectangles $A \times B$ etc., we omit "$\times$"; and we use $A = \{0, 1\}$, $B = \{1, 2\}$. Then K consists of $AAAA$, $AAAB$, $AABB$, $ABBB$, $BBBB$ and $AB1A$, $BB1A$, $B1AB$, $B1AA$. It is a medium. The section "$x_4 = 2$" is $\mathbf{3}^3$ minus two skew lines, edges of the cube, thus a medium; the section "$x_4 = 1$" is the same, and the section "$x_4 = 0$" is a proper ideal in that.

The ideal I in K defined by $x_2 \geqslant 1$, $x_3 \leqslant 1$, consists of eight 3-cells. ($AABB$ meets I only in a 2-cell contained in two of the 3-cells.) As a medium, it is $\mathbf{3}^3$; the functions x_1, $x_2 + x_3 - 1$, x_4 on $I \subset \mathbf{3}^4$ establish an isomorphism, and let us call them u_1, u_2, u_3. In the copies K^1, K^2, K^3 of K that we take next, coordinates are u_j^i ($i, j \in \{1, 2, 3\}$).

We amalgamate the K^i along I, but match respective coordinates u_1^1 to u_2^2 to u_3^3 and so on cyclically. The resulting medium M has, evidently, fifteen 4-cells. No 3-cell is maximal; for instance, $AB1A^1$ is identified with $A1AB^2$ and contained in $AAAB^2$.

It will suffice to show that M is not an amalgam of two ideals E, F, all of whose maximal blocks are 4-cells, with no common 4-cell. Note the symmetry group S_3; besides the obvious Z_3, there are involutions reversing everything (order in $\mathbf{3}$, order of coordinates, cyclic order of superscripts).

Neither of such ideals E, F, can contain a whole K, say K^1. For it would then contain I^2 and the 4-cells $AAAB^2$, $ABBB^2$, the only 4-cells containing certain parts of I^2; but no proper ideal in K^2 contains all this, and we get K^2 and likewise K^3 in with K^1.

So E and F divide each K^i; in fact, $AAAA^i$ is in one of them and $BBBB^i$ in the other. Take Case 1: $AAAA^1 \subset E$, $AAAB^1 \subset F$. Since F is an ideal not containing $AAAA^1$, not F (but E) contains $AB1A^1$, $BB1A^1$, and $B1AA^1$. E cannot contain another maximal block of K^1; for the others end "B" and would put $AB1B^1$ in E, whence the only 4-cell containing it, $ABBB^1$, and $BBBB^1$ and K^1. We find that the maximal cells of K^1 ending "A" are in E, the others in F. Turn to K^3. Here F contains $BB1A^3 = AB1B^1$, similarly $AB1B^3$, hence $ABBB^3$ and (ideal) $BBBB^3$. So $AAAA^3 \subset E$. Also $AB1A^3 \subset F$. Say Subcase 1(a) is $AABB^3 \subset F$. The last two items give $AA1A^3 \subset F$ although E contains $AAAA^1$ and $AAAA^3$. Therefore, F must contain $AAAA^2$. It also contains $B1AB^2$ ($= AB1B^1$), hence $AAAB^2$. But

$A1AB^2$ is $AB1A^1 \subset E$, with the only 4-cell of M containing it locked in F; the subcase is impossible.

We have $AABB^3 \subset E$. With $B1AA^3$, E picks up $B1BB^3$. But F contains $BBBB^1$ and $BBBB^3$; so $BBBB^2 \subset E$. Therefore $AAAA^2 \subset F$. Also $B1AB^2$ $(= AB1B^1)$, hence $AAAB^2 \subset F$. But $A1AB^2$ is in E, a contradiction; Case 1 impossible. Thus $AAAA^1$ and $AAAB^1$ are together, say in E. Take Case 2: $AABB^1 \subset F$. Then $B1AB^1$ is not there (bracketing $AAAB^1$), but in E. If E also contained $AB1A^1$, that would bracket $AB1B^1$, whence $ABBB^1 \subset E$; with $B1AB^1$, $K^1 \subset E$, a contradiction. We conclude $AB1A^1 \subset F$. Also $BBBB^1$ and hence $BB1A^1$. With $AABB^1$, also $AA1A^1$. So, lacking $AAAA^1$, F needs $AAAA^2$ or $AAAA^3$. Not $AAAA^2$; for cycling shows $AB1B^2 \subset F$, hence $ABBB^2$, and also $B1AB^2$ bracketing $BBBB^2$. But $AAAA^3$ is no better. It puts $BBBB^3$ in E, with $A1AB^3$ and thus $B1AB^3$. But there is only one 4-cell on $B1AB^3 = BB1A^1 \subset F$. Case 2 is demolished; and, using symmetry, the proof is complete. (We showed $AAAA^1$ must be with more than half of the 4-cells of K^1.)

(xlix) One can imitate 6.7 to show that a nonempty finite isotropic medium is indecomposable if and only if it is an n-cell for some $n \geqslant 0$, and if and only if it is a product of edges. For finite lattices, on the other hand, indecomposability as media is as far as I can see a somewhat accidental feature. Of course complemented modular lattices are indecomposable. So is any M that has three elements x_1, x_2, x_3 such that $[x_i x_j] = M$ whenever $i \neq j$–e.g. the lattice of subgroups of $Z_4 \times Z_4$.

We conclude with a description of the dual category to symmetric media and some closely related matters. First observe that for a general symmetric medium M, the set M^* of homomorphisms $M \to \mathbf{2}$, taken concretely as a subset of $\mathbf{2}^M$, determines the structure of M (by embedding M in $\mathbf{2}^M$). There are applicable general theorems leading from any such situation to the dual category **[8]**. Of course, some calculations are still required; and it will be simpler to begin more or less ad hoc. Naturally, the question is, what is the structure on the sets M^* induced by the pairing $M \leftrightarrow M^*$?

We define a *binary message* as a partially ordered set with least element 0 and a unary operation x' reversing order, satisfying $x'' = x$ and, if $x \leqslant y$ and $x \leqslant y'$, then $x = 0$. A *morphism* of binary messages is a monotonic mapping preserving the operations 0 and x'. The analogue of (xl) is:

(l) An order ideal I in a binary message $B \neq \{0\}$ maximal subject to containing no elements x and x' contains either x or x', for each x in B. Thus $I = h^{-1}(0)$ for some morphism h: $B \to \mathbf{2}$.

For the ideal join of I and x is just the union J of I with the predecessors of x. If I does not contain x', it contains no successor of x'. Also $x \neq 0'$; so no elements y and y' precede x, and J contains no two y, y'.

Call an ideal *proper* if it contains no two elements y, y'. This high-handed use of a nearly standard word, "proper", can be defended after we note

6.10. THEOREM. *A binary message is embeddable in a power of* $\mathbf{2}$ *and every proper ideal is an intersection of maximal proper ideals.*

PROOF. The latter assertion follows from Zorn's Lemma. Applying it to principal ideals, the preceding lemma gives us the theorem.

It follows that in binary messages, proper ideals are fibers. Obviously, fibers are proper ideals. Thus the order ideals which contain some complementary pair y, y' are "improper" not in the set-theoretic sense but in the sense of incompatibility with the message structure.

Now the medium-message duality turns on the McLuhan object $\mathbf{2}$, which is a medium and a message. It is essential that the two structures are compatible. Accidentally, more is true; and we prove that now–it will serve to recall the meaning of compatibility of an algebraic structure with another structure. A ternary algebra *in* a category $\mathcal{C}$ consists of an object X of $\mathcal{C}$ and a morphism t: $X^3 \to X$ of $\mathcal{C}$; one says that the operation t is *compatible* with the structure of X in $\mathcal{C}$, meaning merely that it is a morphism.

6.11. *A symmetric medium in the category of binary messages is a Boolean algebra with the operations* 0, x', (x, y, z); *the message order is the Boolean order. Every Boolean algebra is a symmetric medium in the binary messages, and the morphisms are the same.*

PROOF. Such a medium B on a message B_0 is, as medium, an interval $[00']$; for the operation (x, y, z) is monotonic, so the relations $\langle 0, x, x\rangle \leqslant \langle 0, x, 0'\rangle \leqslant \langle x, x, 0'\rangle$ yield $(0, x, 0') = x$. The lattice infimum $(0, x, y)$ is also an infimum in B_0. For it is below $x = (x, x, y)$ and $y = (y, x, y)$, while $p \leqslant x$ and $p \leqslant y$ yield $p = (0, p, p) \leqslant (0, x, y)$. So the orderings agree. Order-complements x' make the lattice medium a Boolean algebra, and since complements are unique, the entire message structure is induced by the Boolean structure. The rest of the proof is evident.

6.12. *The categories of finite symmetric media and of finite binary messages are dual.*

PROOF. To a symmetric medium (finite or not) M we associate the set $M^* \subset \mathbf{2}^M$ of homomorphisms $M \to \mathbf{2}$. M^* is a submessage of $\mathbf{2}^M$, for it is closed under pointwise complementation and 0. Moreover, for a homomorphism f: $M \to N$ the mapping $f^* = \mathrm{Hom}(f, \mathbf{2})$: $N^* \to M^*$ is a message morphism, since the operations and order are defined pointwise. Of course M^* is finite when M is, so we have a functor $H = \mathrm{Hom}(\ , \mathbf{2})$ from finite symmetric media to finite binary messages. Similarly we have contravariant $G = \mathrm{Hom}(\ , \mathbf{2})$ in the opposite direction; $G(A)$ is a submedium of 2^A since (x, y, z) preserves order, complementation and 0, and G is functorial for the same reason. Evaluation induces an embedding $M \to GH(M)$ natural in M. But every point p of $GH(M)$, i.e. every message morphism p: $M^* \to \mathbf{2}$, is actually given by a point p_0 of M; that is, $p(f) = 1$ if and only if $f(p_0) = 1$. For every two of the ideals $f^{-1}(1)$ (for which $p(f) = 1$) have a common point since p preserves order and complementation; hence by 1.1, all have a common point p_0, and it is unique since M^* separates points of M. Thus GH is naturally equivalent to the identity.

In the same way, evaluation embeds messages A in $HG(A)$ naturally. Consider a homomorphism q: $G(A) \to \mathbf{2}$, i.e. a betweenness-preserving mapping. The elements j: $A \to \mathbf{2}$ of $G(A)$ are determined by their kernels J, which are maximal (proper) ideals. In $G(A)$, note: l is between j and k iff $L \supset J \cap K$ iff $L \subset J \cup K$. Let Q be the intersection of all J for which $q(j) = 1$. Since Q is a finite intersection and 5.12 applies, $q(k) = 1$ if and only if $K \supset Q$. It remains to check that Q must be principal. Otherwise it could be covered by maximal ideals J, K not containing Q. $Q \cup (J \cap K)$ would be a proper ideal I, and a maximal ideal containing I is necessarily between J and K and contradictory. This shows that $A \to HG(A)$ is bijective. It is isomorphic since $x \leqslant y$ if every maximal ideal containing y contains x; and the lemma is proved.

From 6.12, general duality theory [**8**] takes over. So we can state the results; but since direct proof similar to 6.12 is not hard, we shall sketch that, and use this as an excuse for not introducing the terminology of [**8**]. A *compact space* in a category is an instance of an algebra in a category; the algebraic operations are highly infinitary (see e.g. [**9**]), but if the category is complete there is no difficulty. A *profinite* object in a category with a distinguished forgetful functor to sets is an inverse limit of objects whose underlying sets are finite. From 6.12, the finiteness of the object $\mathbf{2}$ representing the duality, and the finiteness of all finitely generated symmetric media and also, finitely generated binary messages, by [**8**] we have

6.13. THEOREM. *The categories of symmetric media and of profinite compact binary messages are dual. The categories of binary messages and of profinite compact symmetric media are dual.*

SKETCH OF DIRECT PROOF. For a symmetric medium M, $M^* \subset \mathbf{2}^M$ has a topology induced by the product topology. Since a pointwise limit of homomorphisms is a homomorphism, M^* is compact; and it is the inverse limit of its projections in finite partial products. Since M is the direct limit of its finite submedia M_i, we can use the $\mathbf{2}^{M_i}$ to show that M^* is a profinite compact message. So we get a functor H as before, and similarly G on these profinite messages to discrete media. To show that $M \to GH(M)$ is surjective, observe that the continuous message morphisms p: $M^* \to \mathbf{2}$ are determined by the compact open sets $p^{-1}(0)$, $p^{-1}(1)$. M^* has a basis of open sets which are fibers of projections $M^* \to \mathbf{2}^{M_i}$, and $p^{-1}(0)$ and $p^{-1}(1)$ are finite unions of those; so we descend to the finite case and have the result. Again, for the map $A \to HG(A)$ of compact objects to be surjective, we need only approximate each q: $G(A) \to \mathbf{2}$ by evaluation at a point of A, and the finite case does it. For binary messages and suitable compact media, the proof is essentially the same.

Apart from abstract nonsense, what are these profinite compact objects? Amusingly,

6.14. *Every compact symmetric medium whose underlying compact space is profinite is profinite.*

But

6.15. *Proposition* 6.14 *is not true in binary messages.*

Let us finish the recital of facts; proofs last. Having 6.14, there seems to be no danger of confusion in describing those Boolean-space media as *Boolean media*: the topological media which are closed subobjects of powers of **2**.

6.16. *Every closed ideal in a Boolean medium is a fiber of a continuous homomorphism to a Boolean medium.*

I do not know whether 6.16 is true in profinite compact binary messages. Finally, a universal problem (rather closely related to the problem of calculating the theory of a dual) solved by an extension of symmetric medium theory:

6.17. *Every operation of complete atomic Boolean algebras, f: $\mathbf{2}^I \to \mathbf{2}$, which commutes with complementation is definable in terms of median.*

Proof of 6.14. Any open covering of such a medium M is refined by a finite partition into open-closed sets U_i. Consider the relation R in $M \times M$ consisting of those $\langle p, q\rangle$ such that for all x and y, (p, x, y) is in the same U_i as (q, x, y). Evidently R is an equivalence relation. By $(*)$ and symmetry, $((p, x, y), u, v) = (p, (x, u, v), (y, u, v))$; so $\langle (p, x, y), (q, x, y)\rangle \in R$ when $\langle p, q\rangle \in R$. As (q, x, y) is (further) R-equivalent to (q, x^*, y) if $\langle x, x^*\rangle \in R$, and likewise for y, y^*, thus R is a congruence relation. By a standard compactness argument (reflecting the equicontinuity of (p, x, y) as $f_p(x, y)$), the R-classes are open and closed. So M/R is finite.

Proof of 6.15. Take two ordinary Cantor sets C, C'. Identify 0 in C with $1'$ in C', and 1 in C with $0'$ in C'. Extend the definition of x' over the resulting "Cantor circle" in the obvious way. We define an ordering which relates each point $x \neq 0, 1$ to at most three other points. Of course 0 is least and 1 is greatest. Also, for each interval (pq) removed in the "middle third" construction of C, put $p \geqslant q$ and $p' \leqslant q'$. We have a closed partial order with respect to which x' is complementing and order-reversing, and 0 is least; so we have a compact message on a Boolean space. But there is no open-closed proper ideal I. For if I is an open-closed set containing 0 but not 1, $I \cap C$ has a largest point p in the usual order of C, and p is a left endpoint of a removed interval; so I is not an ideal in the present order.

Proof of 6.16. In media, the image of an ideal under a surjective homomorphism h is an ideal, for if $h(x)$ is between $h(y)$ and $h(z)$ it is equal to $h((x, y, z))$. So a closed ideal I in an inverse limit of finite symmetric media, being the intersection of the inverse images of its projections I_α, which are fibers, is a fiber.

Remark. Apropos 6.16, I do not know if the smallest congruence relation R having the closed ideal I in one congruence class gives a Boolean quotient. It is easy to verify that it gives a compact quotient; R is the set of all $\langle p, q\rangle$ such that for some i, j in I, $p \in [iq]$ and $q \in [jp]$, which is a closed set.

Apropos the problem just raised, there is a more geometric construction for a small open-closed ideal V containing a given closed ideal I. In any open-closed neighborhood U of I, let W be the (starshaped) set of all $x \in U$ such that $[xi] \subset U$ for all $i \in I$; let V be the set of $x \in W$ such that $[xw] \subset W$ for all $w \in W$.

PROOF OF 6.17. Complementation itself is definable, $y = x'$ by $(\forall z)(x, y, z) = z$. Now recall that finitary Boolean operations are expressible in terms of 0, x', and (x, y, z); rephrasing, any Boolean $f(x_1, \ldots, x_n)$ is $\varphi(x_1, \ldots, x_n, 0)$ where φ is an expression in terms of complements and medians. If f commutes with complementation of its n arguments, while φ commutes with complementation of its $n + 1$ arguments, we have

$$\varphi(x_1, \ldots, x_n, 1) = \varphi(x_1', \ldots, x_n', 0)' = f(x_1', \ldots, x_n')' = f(x_1, \ldots, x_n).$$

Thus the identity $f(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_n, x_1)$ is valid in the Boolean algebra **2**; and hence it is valid. (This does not apply for $n = 0$, but then there are no such f.)

The proof is substantially the same for infinite n. Every function $\mathbf{2}^I \to \mathbf{2}$ is expressible as an infinite join of characteristic functions of points; those are infinite meets of coordinate functions and their complements. The joins and meets are definable in terms of median if an extra argument 0 is introduced. Finally, if the composite is a definition of an operation (or, for that matter, relation) commuting with complementation, 0 can be eliminated in the same way as before.

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